LECTURE NOTES

ON

MATHEMATICS III
ACADEMIC YEAR 2021-22

I B.Tech –II SEMESTER (R20)

P.S.S.Srilatha, Assistant Professor

DEPARTMENT OF HUMANITIES AND BASIC SCIENCES

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DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING

I Year II Semester		L	T	P	C
		3	0	0	3
MATHEMATICS-III(Vector Calculus, Transforms and PDE)					

Course Objectives:

- To familiarize the techniques in partial differential equations
- To furnish the learners with basic concepts and techniques at plus two level to lead them intoadvanced level by handling various real-world applications.

Course Outcomes: At the end of the course, the student will be able to

- interpret the physical meaning of different operators such as gradient, curl and divergence(L5)
- estimate the work done against a field, circulation and flux using vector calculus (L5)
- apply the Laplace transform for solving differential equations (L3)
- find or compute the Fourier series of periodic signals (L3)
- know and be able to apply integral expressions for the forwards and inverse Fouriertransform to a range of non-periodic waveforms (L3)
- identify solution methods for partial differential equations that model physical processes(L3)

UNIT –I: Vector calculus: (10 hrs)

Vector Differentiation: Gradient– Directional derivative – Divergence– Curl– Scalar Potential

Vector Integration: Line integral – Work done – Area– Surface and volume integrals – Vector integral theorems: Greens, Stokes and Gauss Divergence theorems (without proof) and problems on above theorems.

UNIT –II: Laplace Transforms: (10 hrs)

Laplace transforms – Definition and Laplace transforms of some certain functions– Shifting theorems – Transforms of derivatives and integrals – Unit step function – Dirac's delta function Periodic function – Inverse Laplace transforms– Convolution theorem (without proof).
Applications: Solving ordinary differential equations (initial value problems) using Laplace transforms.

UNIT –III: Fourier series and Fourier Transforms: (10 hrs)

Fourier Series: Introduction– Periodic functions – Fourier series of periodic function – Dirichlet's conditions – Even and odd functions –Change of interval– Half-range sine and cosine series.

Fourier Transforms: Fourier integral theorem (without proof) – Fourier sine and cosine integrals – Sine and cosine transforms – Properties (article-22.5 in text book-1nverse transforms – Convolution theorem (without proof) – Finite Fourier transforms.

UNIT –IV: PDE of first order: (8 hrs)

Formation of partial differential equations by elimination of arbitrary constants and arbitrary functions – Solutions of first order linear (Lagrange) equation and nonlinear (standard types)equations.

UNIT – V: Second order PDE and Applications: (10 hrs)

Second order PDE: Solutions of linear partial differential equations with constant coefficients Nonhomogeneous term of *type* $e^{ax+by}, \sin(ax+by), \cos(ax+by), x^m y^n$.

Applications of PDE: Method of separation of Variables– Solution of One-dimensional Wave, Heatand two-dimensional Laplace equation.

Text Books:

1. B. S. Grewal, Higher Engineering Mathematics, 44th Edition, Khanna Publishers, 2018.

B. V. Ramana, Higher Engineering Mathematics, 2007 Edition, Tata McGraw Hill Education

Reference Books:

1. Erwin Kreyszig, Advanced Engineering Mathematics, 10th Edition, Wiley-India. 2015.
2. Dean. G. Duffy, Advanced Engineering Mathematics with MATLAB, 3rd Edition, CRCPress, 2010.
3. Peter O' Neil, Advanced Engineering Mathematics, 7th edition, Cengage, 2011..

Srimantha Pal, S C Bhunia, Engineering Mathematics, Oxford University Press, 2015

VSM COLLEGE OF ENGINEERING

RAMACHANDRAPRUM-533255

DEPARTMENT OF HUMANITIES & BASIC SCIENCES

Course Title	Year-Sem	Branch	Contact Periods/Week	Sections
MATHEMATICS-III	1-II	EEE	6	A

- **COURSE OUTCOMES:** Students are able to
- Interpret the physical meaning of different operators such as gradient, curl and divergence (L5)
- Estimate the work done against a field, circulation and flux using vector calculus (L5)
- Apply the Laplace transform for solving differential equations (L3)
- Find or compute the Fourier series of periodic signals (L3)
- Know and be able to apply integral expressions for the forwards and inverse Fourier transform to a range of non-periodic waveforms (L3)
- Identify solution methods for partial differential equations that model physical processes (L3)

Unit/ item No.	Outcomes	Topic		Number of periods	Total periods	Book Reference	Delivery Method
1	**CO2:** Interpret the different operators such as gradient, curl and divergence		**Vector calculus**		**18**	T1,T3 , R2	Chalk & Talk, Active Learning & tutorial
		1.1	Vector Differentiation-introduction	2			
		1.2	Gradient– Directional derivative	2			
		1.3	Divergence– Curl– Scalar Potential	3			
	CO5: Estimate the work done against a field, circulation and flux using vector calculus	1.4	Vector Integration: Line integral, Work done – Area– Surface and volume integrals	5			
		1.5	Vector integral theorems: Greens, Stokes and Gauss Divergence theorems (without proof) and	6			

			problems on above theorems.				
2	**CO3:** apply the Laplace transform for solving differential equations		**Laplace Transforms**				
		2.1	Definition and Laplace transforms of some certain functions	3	**15**		
		2.2	Shifting theorems Transforms of derivatives and integrals – Unit step function	4			
		2.3	.Dirac's delta function Periodic function-Inverse Laplace transforms–Convolution theorem (without proof)	4			
		2.4	Applications: Solving ordinary differential equations (initial value problems) using Laplace transforms	4			
3	**CO4:** Analyze and compute the Fourier series of periodic signals		**Fourier series and Fourier Transforms**		**18**	T1,T3, T4,R4	Chalk & Talk
		3.0	Fourier Series: Introduction– Periodic functions	2			
		3.1	Fourier series of periodic function	2			
		3.2	Dirichlet's conditions – Even and odd functions	3			
		3.3	Change of interval– Half-range sine and cosine series	2			
		3.4	Fourier Transforms: Fourier integral theorem (without proof	3			
		3.5	Fourier sine and cosine integrals – Sine and cosine transforms – Properties	3			
		3.6	inverse transforms Convolution theorem (without proof) Finite Fourier transforms.	3			
			PDE of first order				
4	. **CO1:** Gain the Knowledge on solving first order partial differential equations.	4.1	Formation of partial differential equations by elimination of arbitrary constants and arbitrary functions	**5**	**11**	T1,T2, T3, R1,R2	Chalk & Talk,
		4.2	Solutions of first order linear (Lagrange) equation and nonlinear (standard types) equations.	**6**			

			Second order PDE and Applications				
5	**CO3:** Find out the solution and apply those solution methods for partial differential equations that model physical processes	5.1	Second order PDE: Solutions of linear partial differential equations with constant coefficients	**5**	**10**		
		5.2	Non-Homogeneous type e^{ax+by} ,sin($ax + by$), cos($ax + by$), $x^m y^n$	**5**			Chalk & Talk,
			TOTAL	**72**			

LIST OF TEXT BOOKS AND AUTHORS

Text Books:

1. **B. S. Grewal,** Higher Engineering Mathematics, 44th Edition, Khanna Publishers.
2. **B. V. Ramana,**Higher Engineering Mathematics, 2007 Edition, Tata Mc. Graw Hill Education.

Reference Books:

1. **Erwin Kreyszig,**Advanced Engineering Mathematics, 10th Edition, Wiley-India.
2. **Dean. G. Duffy,**Advanced Engineering Mathematics with MATLAB, 3rd Edition, CRC Press.
3. **Peter O' Neil,** Advanced Engineering Mathematics, Cengage.
4. **Srimantha Pal, S C Bhunia,** Engineering Mathematics, Oxford University Press.

Faculty Member **Head of the Department** **Principal**

UNIT-I

* Vector Calculus *

Vector differentiation:

Vector: A vector is a physical Quantity which has both magnitude and direction is called as vector.

Ex:- Velocity, Accelaration, Force etc.

Scalar: A physical Quantity which has only magnitude is called scalar.

Ex:- mass, length, area etc

Unit Vector:- A vector which of length 1 unit is called as a unit vector. If $\bar{a}$ is any non zero vector then $\frac{\bar{a}}{|\bar{a}|}$ is called as unit vector in the direction of $\bar{a}$.

It is denoted by $\hat{a}$.

position vector:- let 'o' be the fixed point in the space called as origin. If p is any point in the space then $\overline{op}$ is called position vector of p w.r.t 'o'.

If A, B are two points in the space then $[\overline{AB} = \overline{OB} - \overline{OA}]$

If r is the position vector of the point $p(x,y,z)$ then

$r = x\bar{i} + y\bar{j} + z\bar{k}$

Note:- If $\bar{a} = x_1\bar{i} + y_1\bar{j} + z_1\bar{k}$; $\bar{b} = x_2\bar{i} + y_2\bar{j} + z_2\bar{k}$

$\bar{a}\cdot\bar{b} = (x_1\bar{i} + y_1\bar{j} + z_1\bar{k})\cdot(x_2\bar{i} + y_2\bar{j} + z_2\bar{k})$

$\bar{a}\cdot\bar{b} = x_1x_2 + y_1y_2 + z_1z_2$

$$\bar{a}\times\bar{b} = \begin{bmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}$$

$$\bar{a}\times\bar{b} = \bar{i}(y_1z_2 - y_2z_1) - \bar{j}(x_1z_2 - x_2z_1) + \bar{k}(x_1y_2 - x_2y_1)$$

Vector differentiation:-

let $f(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$ $\overset{f_1(t), f_2(t), f_3(t)}{\times}$ are real valued functions.

Derivative:- let $\bar{f}$ be a vector function on an interval I and $a \in I$ then $lt_{t\to a} \dfrac{\bar{f}(t) - \bar{f}(a)}{t-a}$ if exists is called the derivative of $\bar{f}$ at a, and is denoted by $f'(a)$ or $\left[\dfrac{d\bar{f}}{dt}\right]$ at $t = a$. we also say that $\bar{f}$ is differentiable at $t = a$ if $f'(a)$ exists.

$$\therefore \frac{d}{dt}[f(t)] = \frac{d}{dt}[f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}]$$

$$\frac{df}{dt} = f_1'(t)\bar{i} + f_2'(t)\bar{j} + f_3'(t)\bar{k}$$

$$\frac{d^2f}{dt^2} = f''(t)\bar{i} + f_2''(t)\bar{j} + f_3''(t)\bar{k}$$

Scalar & Vector point functions:

Consider a reasonin region in 3d space to each point $p(x,y,z)$ suppose we assosicate a unique real number (called scalar) say ϕ. This $\phi(x,y,z)$ is called a scalar point function defined on the region. Simillarly into each point $p(x,y,z)$ we assosiate a unib vector $\bar{f}(x,y,z)$, $\bar{f}$ is called a vector point function.

Ex:- Take a Heated solid at each point $p(x,y,z)$ of the solid there will be temperature (T). This (T) is scalar point function.

Ex(I):- consider a particle moving in a space at each point p on its path the particle will having a velocity $\bar{v}$ which is a vector point function

Vector differential operator:

It is denoted by ∇ (del); and it is defined as

$$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$$

Gradient of scalar point function:

let $\phi(x,y,z)$ be a scalar point function the gradient of ϕ is denoted by $grad\phi$ or $\nabla\phi$. And is defined by

$$\nabla\phi = \left(\bar{i}\cdot\frac{\partial}{\partial x} + \bar{j}\cdot\frac{\partial}{\partial y} + \bar{k}\cdot\frac{\partial}{\partial z}\right)\phi$$

$$\nabla\phi = \bar{i}\cdot\frac{\partial\phi}{\partial x} + \bar{j}\cdot\frac{\partial\phi}{\partial y} + \bar{k}\cdot\frac{\partial\phi}{\partial z}$$

Properties:-

If f and g are two scalar point functions then the

$grad(f \pm g) = grad f + grad g$

$grad(fg) = f grad g + g grad f$

$grad(f/g) = \dfrac{g grad f - f grad g}{g^2}$ $(\because g \neq 0)$

If f is a scalar point function and c is a constant then $grad(cf) = c \cdot grad(f)$

Tangent Vector to a Curve :-

let $\bar{r} = F(t)$ be a vector curve then $\frac{d\bar{r}}{dt} \Rightarrow$ is the Tangent of the curve where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and x, y, z are functions of t, then $\boxed{\frac{d\bar{r}}{dt} = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}}$

The unit tangent vector is $\frac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|}$ where $\frac{d\bar{r}}{dt}$ represents a vector along tangent to the curve.

Normal vector to the surface :-

let $\phi(x, y, z) = k$ be the surface then the normal vector to the surface is equal to gradϕ. The unit normalvector to the surface is $\frac{\nabla\phi}{|\nabla\phi|}$

Angle between two surfaces :-

let $f(x, y, z) = k$ and $\phi(x, y, z) = c$ be the two surfaces then the angle between two surfaces at a point P is the angle between the normals to the given surface at that point. let θ be the angle between two normal vectors then cos

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1|\,|\bar{n}_2|}$$

Directional Derivative :-

let $\phi(x, y, z)$ is a scalar point function then the directional derivative of ϕ at point P in the direction of $\bar{n}$ is defined by

$$\text{Directional derivative} = [\text{grad}\,\phi]_P = \frac{\bar{n}}{|\bar{n}|}$$

where $\hat{e} = \frac{\bar{n}}{|\bar{n}|}$; $\hat{e}$ is the unit vector in the direction of $\bar{n}$.

$$\text{directional derivative} = \nabla\phi \cdot \hat{e}$$

Note :- Greatest value of directional derivative of ϕ at a point P is equal to $|\text{grad}\,\phi| = |\nabla\phi|$

1) Find Gradϕ where $\phi = x^3 + y^3 + 3xyz$?

Given :- $\phi = x^3 + y^3 + 3xyz$

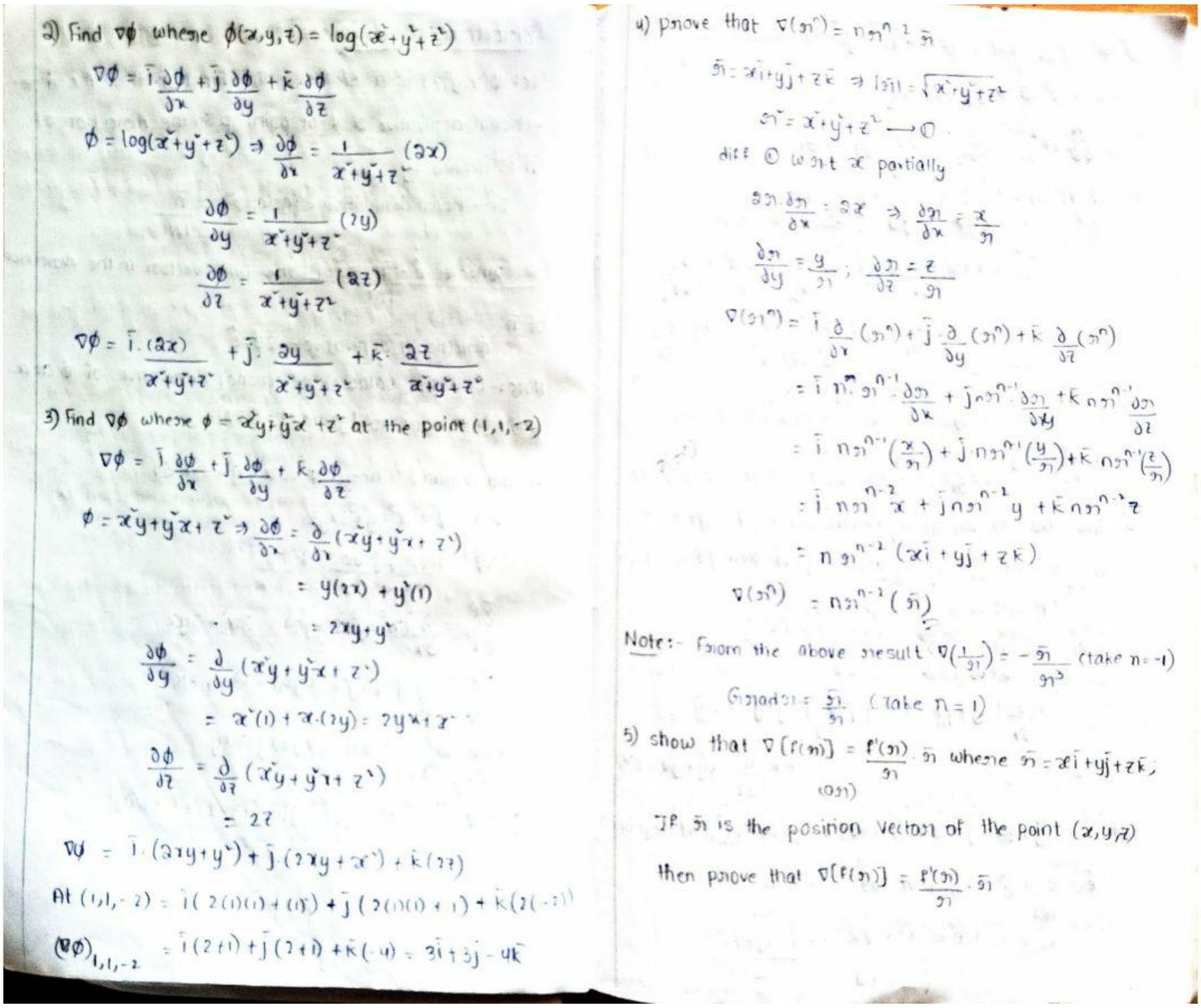
$$\nabla\phi = \bar{i} \cdot \frac{\partial\phi}{\partial x} + \bar{j} \cdot \frac{\partial\phi}{\partial y} + \bar{k} \cdot \frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}(x^3 + y^3 + 3xyz) = 3x^2 + 3yz$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}(x^3 + y^3 + 3xyz) = 3y^2 + 3xz$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial}{\partial z}(x^3 + y^3 + 3xyz) = 3xy$$

$$\text{grad}\,\phi = \bar{i} \cdot (3x^2 + 3yz) + \bar{j}(3y^2 + 3xz) + \bar{k}(3xy)$$



2) Find $\nabla\phi$ where $\phi(x,y,z) = \log(x^2+y^2+z^2)$

$$\nabla\phi = \bar{i}\,\frac{\partial\phi}{\partial x} + \bar{j}\,\frac{\partial\phi}{\partial y} + \bar{k}\,\frac{\partial\phi}{\partial z}$$

$$\phi = \log(x^2+y^2+z^2) \Rightarrow \frac{\partial\phi}{\partial x} = \frac{1}{x^2+y^2+z^2}(2x)$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{x^2+y^2+z^2}(2y)$$

$$\frac{\partial\phi}{\partial z} = \frac{1}{x^2+y^2+z^2}(2z)$$

$$\nabla\phi = \bar{i}.\frac{(2x)}{x^2+y^2+z^2} + \bar{j}.\frac{2y}{x^2+y^2+z^2} + \bar{k}.\frac{2z}{x^2+y^2+z^2}$$

3) Find $\nabla\phi$ where $\phi = x^2y + y^2x + z^2$ at the point $(1,1,-2)$

$$\nabla\phi = \bar{i}\,\frac{\partial\phi}{\partial x} + \bar{j}\,\frac{\partial\phi}{\partial y} + k\,\frac{\partial\phi}{\partial z}$$

$$\phi = x^2y+y^2x+z^2 \Rightarrow \frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}(x^2y+y^2x+z^2)$$

$$= y(2x) + y^2(1)$$

$$= 2xy + y^2$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}(x^2y+y^2x+z^2)$$

$$= x^2(1) + x.(2y) = 2yx + x^2$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial}{\partial z}(x^2y+y^2x+z^2)$$

$$= 2z$$

$$\nabla\phi = \bar{i}.(2xy+y^2) + \bar{j}.(2xy+x^2) + \bar{k}(2z)$$

At $(1,1,-2) = \bar{i}(2(1)(1)+(1)^2) + \bar{j}(2(1)(1)+1) + \bar{k}(2(-2))$

$$(\nabla\phi)_{1,1,-2} = \bar{i}(2+1) + \bar{j}(2+1) + \bar{k}(-4) = 3\bar{i} + 3\bar{j} - 4\bar{k}$$

4) prove that $\nabla(r^n) = nr^{n-2}\,\bar{r}$

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow |\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2 \longrightarrow ①$$

diff ① w.r.t x partially

$$2r.\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r};\quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla(r^n) = \bar{i}\,\frac{\partial}{\partial x}(r^n) + \bar{j}\,\frac{\partial}{\partial y}(r^n) + \bar{k}\,\frac{\partial}{\partial z}(r^n)$$

$$= \bar{i}\,n.r^{n-1}.\frac{\partial r}{\partial x} + \bar{j}\,nr^{n-1}.\frac{\partial r}{\partial y} + \bar{k}\,nr^{n-1}\frac{\partial r}{\partial z}$$

$$= \bar{i}\,nr^{n-1}\left(\frac{x}{r}\right) + \bar{j}\,nr^{n-1}\left(\frac{y}{r}\right) + \bar{k}\,nr^{n-1}\left(\frac{z}{r}\right)$$

$$= \bar{i}\,nr^{n-2}x + \bar{j}\,nr^{n-2}y + \bar{k}\,nr^{n-2}z$$

$$= n\,r^{n-2}(x\bar{i} + y\bar{j} + z\bar{k})$$

$$\nabla(r^n) = nr^{n-2}(\bar{r})$$

Note:- From the above result $\nabla\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$ (take $n=-1$)

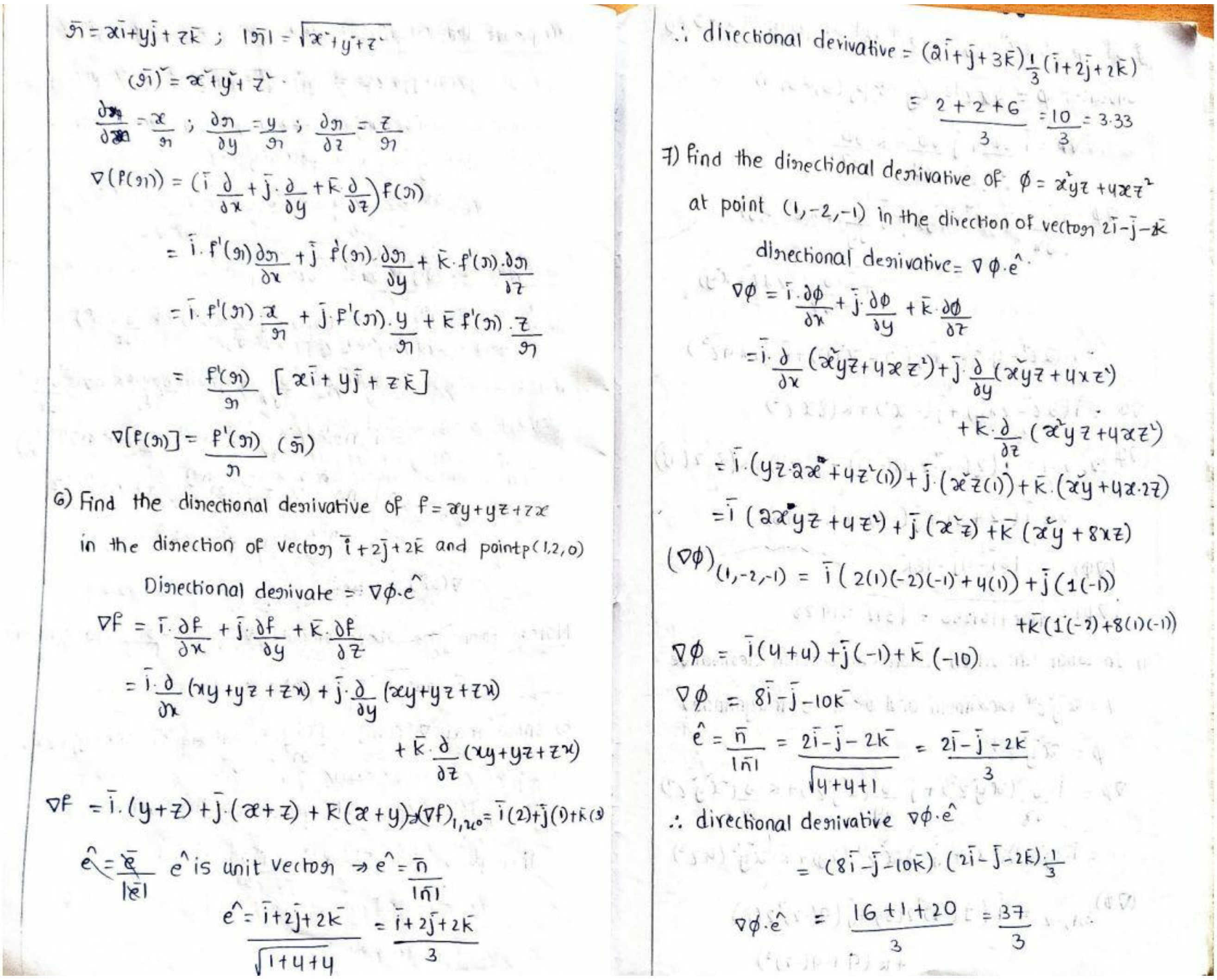
Grad $r = \frac{\bar{r}}{r}$ (Take $n = 1$)

5) show that $\nabla[f(r)] = \frac{f'(r)}{r}\,\bar{r}$ where $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$;

(or)

If $\bar{r}$ is the position vector of the point (x,y,z)

then prove that $\nabla[f(r)] = \frac{f'(r)}{r}.\bar{r}$



$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$; $|\bar{r}| = \sqrt{x^2+y^2+z^2}$

$(\bar{r})^2 = x^2 + y^2 + z^2$.

$\frac{\partial r}{\partial x} = \frac{x}{r}$; $\frac{\partial r}{\partial y} = \frac{y}{r}$; $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla(f(r)) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\cdot\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)f(r)$$

$$= \bar{i}\cdot f'(r)\frac{\partial r}{\partial x} + \bar{j}\ f'(r)\cdot\frac{\partial r}{\partial y} + \bar{k}\cdot f'(r)\cdot\frac{\partial r}{\partial z}$$

$$= \bar{i}\cdot f'(r)\frac{x}{r} + \bar{j}\cdot f'(r)\cdot\frac{y}{r} + \bar{k}\,f'(r)\cdot\frac{z}{r}$$

$$= \frac{f'(r)}{r}\left[x\bar{i} + y\bar{j} + z\bar{k}\right]$$

$$\nabla[f(r)] = \frac{f'(r)}{r}(\bar{r})$$

6) Find the directional derivative of $f = xy + yz + zx$ in the direction of vector $\bar{i} + 2\bar{j} + 2\bar{k}$ and point $p(1,2,0)$

Directional derivate $= \nabla\phi\cdot\hat{e}$

$$\nabla f = \bar{i}\cdot\frac{\partial f}{\partial x} + \bar{j}\cdot\frac{\partial f}{\partial y} + \bar{k}\cdot\frac{\partial f}{\partial z}$$

$$= \bar{i}\cdot\frac{\partial}{\partial x}(xy + yz + zx) + \bar{j}\cdot\frac{\partial}{\partial y}(xy + yz + zx) + \bar{k}\cdot\frac{\partial}{\partial z}(xy + yz + zx)$$

$\nabla f = \bar{i}\cdot(y+z) + \bar{j}\cdot(x+z) + \bar{k}(x+y)$; $(\nabla f)_{1,2,0} = \bar{i}(2) + \bar{j}(1) + \bar{k}(3$

$\hat{e} = \frac{\bar{e}}{|\bar{e}|}$ $\hat{e}$ is unit vector $\Rightarrow \hat{e} = \frac{\bar{n}}{|\bar{n}|}$

$$\hat{e} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1+4+4}} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}$$

∴ directional derivative $= (2\bar{i} + \bar{j} + 3\bar{k})\frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})$

$$= \frac{2+2+6}{3} = \frac{10}{3} = 3.33$$

7) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at point $(1,-2,-1)$ in the direction of vector $2\bar{i} - \bar{j} - 2\bar{k}$

directional derivative $= \nabla\phi\cdot\hat{e}$.

$$\nabla\phi = \bar{i}\cdot\frac{\partial\phi}{\partial x} + \bar{j}\cdot\frac{\partial\phi}{\partial y} + \bar{k}\cdot\frac{\partial\phi}{\partial z}$$

$$= \bar{i}\cdot\frac{\partial}{\partial x}(x^2yz + 4xz^2) + \bar{j}\cdot\frac{\partial}{\partial y}(x^2yz + 4xz^2) + \bar{k}\cdot\frac{\partial}{\partial z}(x^2yz + 4xz^2)$$

$$= \bar{i}\cdot(yz\cdot 2x + 4z^2(1)) + \bar{j}\cdot(x^2z(1)) + \bar{k}\cdot(x^2y + 4x\cdot 2z)$$

$$= \bar{i}\,(2xyz + 4z^2) + \bar{j}\,(x^2z) + \bar{k}\,(x^2y + 8xz)$$

$$(\nabla\phi)_{(1,-2,-1)} = \bar{i}\,(2(1)(-2)(-1) + 4(1)) + \bar{j}\,(1(-1)) + \bar{k}(1(-2) + 8(1)(-1))$$

$$\nabla\phi = \bar{i}(4+4) + \bar{j}(-1) + \bar{k}\,(-10)$$

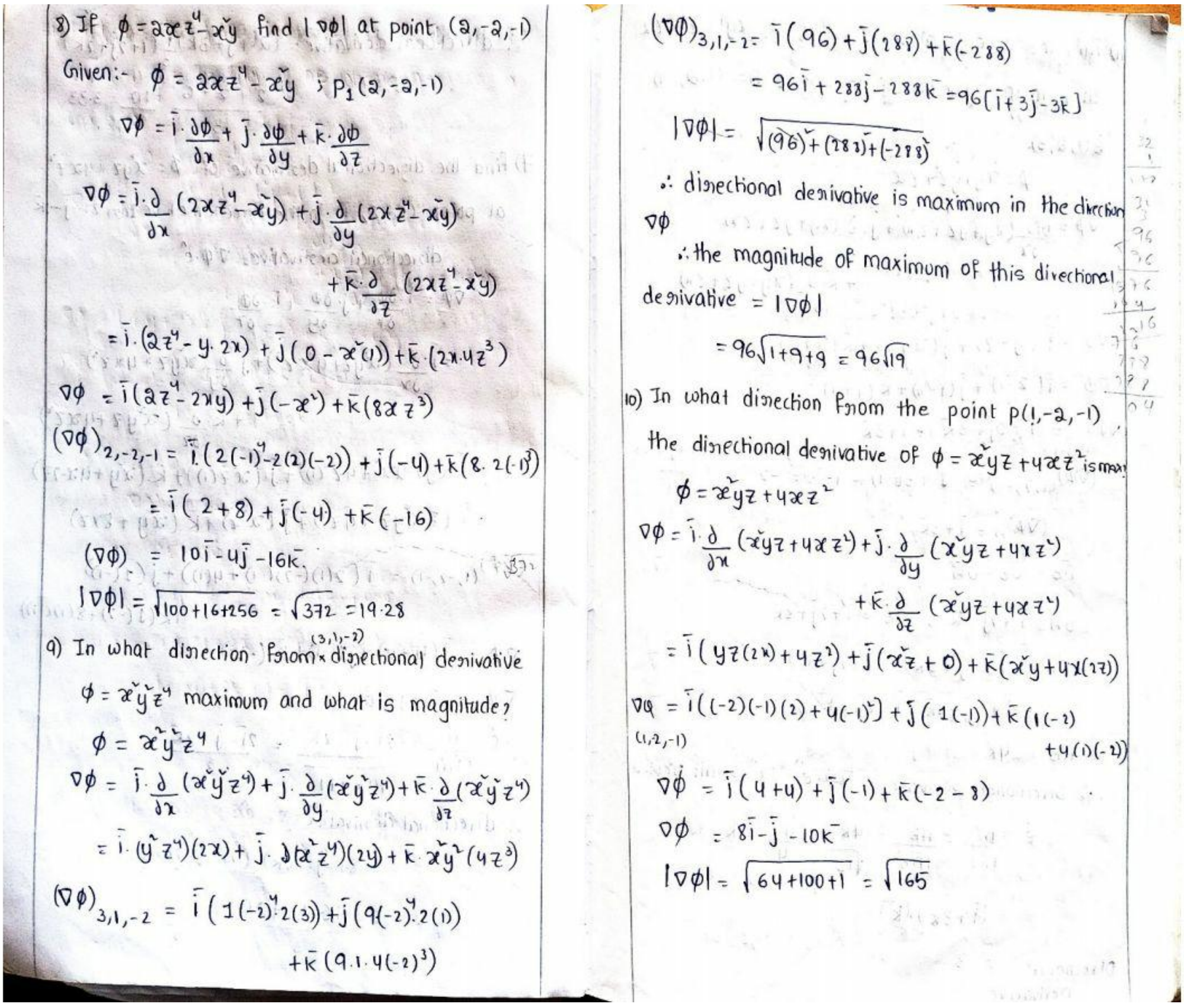
$$\nabla\phi = 8\bar{i} - \bar{j} - 10\bar{k}$$

$$\hat{e} = \frac{\bar{n}}{|\bar{n}|} = \frac{2\bar{i} - \bar{j} - 2\bar{k}}{\sqrt{4+4+1}} = \frac{2\bar{i} - \bar{j} - 2\bar{k}}{3}$$

∴ directional derivative $\nabla\phi\cdot\hat{e}$

$$= (8\bar{i} - \bar{j} - 10\bar{k})\cdot(2\bar{i} - \bar{j} - 2\bar{k})\frac{1}{3}$$

$$\nabla\phi\cdot\hat{e} = \frac{16+1+20}{3} = \frac{37}{3}$$



8) If $\phi = 2xz^4 - x^2y$ find $|\nabla\phi|$ at point $(2,-2,-1)$

Given:- $\phi = 2xz^4 - x^2y$; $P_1(2,-2,-1)$

$$\nabla\phi = \bar{i}\cdot\frac{\partial\phi}{\partial x} + \bar{j}\cdot\frac{\partial\phi}{\partial y} + \bar{k}\cdot\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \bar{i}\cdot\frac{\partial}{\partial x}(2xz^4 - x^2y) + \bar{j}\cdot\frac{\partial}{\partial y}(2xz^4 - x^2y)$$

$$+ \bar{k}\cdot\frac{\partial}{\partial z}(2xz^4 - x^2y)$$

$$= \bar{i}\cdot(2z^4 - y\cdot 2x) + \bar{j}(0 - x^2(1)) + \bar{k}\cdot(2x\cdot 4z^3)$$

$$\nabla\phi = \bar{i}(2z^4 - 2xy) + \bar{j}(-x^2) + \bar{k}(8xz^3)$$

$$(\nabla\phi)_{2,-2,-1} = \bar{i}.(2(-1)^4 - 2(2)(-2)) + \bar{j}(-4) + \bar{k}(8\cdot 2(-1)^3)$$

$$= \bar{i}(2+8) + \bar{j}(-4) + \bar{k}(-16)$$

$$(\nabla\phi) = 10\bar{i} - 4\bar{j} - 16\bar{k}.$$

$$|\nabla\phi| = \sqrt{100+16+256} = \sqrt{372} = 19.28$$

9) In what direction from $\overset{(3,1,-2)}{\times}$ directional derivative $\phi = x^2y^2z^4$ maximum and what is magnitude?

$$\phi = x^2y^2z^4$$

$$\nabla\phi = \bar{i}\cdot\frac{\partial}{\partial x}(x^2y^2z^4) + \bar{j}\cdot\frac{\partial}{\partial y}(x^2y^2z^4) + \bar{k}\cdot\frac{\partial}{\partial z}(x^2y^2z^4)$$

$$= \bar{i}\cdot(y^2z^4)(2x) + \bar{j}\cdot\partial(x^2z^4)(2y) + \bar{k}\cdot x^2y^2(4z^3)$$

$$(\nabla\phi)_{3,1,-2} = \bar{i}(1(-2)^4\cdot 2(3)) + \bar{j}(9(-2)^4\cdot 2(1))$$

$$+ \bar{k}(9.1.4(-2)^3)$$

$$(\nabla\phi)_{3,1,-2} = \bar{i}(96) + \bar{j}(288) + \bar{k}(-288)$$

$$= 96\bar{i} + 288\bar{j} - 288\bar{k} = 96[\bar{i} + 3\bar{j} - 3\bar{k}]$$

$$|\nabla\phi| = \sqrt{(96)^2 + (288)^2 + (-288)^2}$$

∴ directional derivative is maximum in the direction $\nabla\phi$

∴ the magnitude of maximum of this directional derivative $= |\nabla\phi|$

$$= 96\sqrt{1+9+9} = 96\sqrt{19}$$

10) In what direction from the point $P(1,-2,-1)$ the directional derivative of $\phi = x^2yz + 4xz^2$ is max

$$\phi = x^2yz + 4xz^2$$

$$\nabla\phi = \bar{i}\cdot\frac{\partial}{\partial x}(x^2yz + 4xz^2) + \bar{j}\cdot\frac{\partial}{\partial y}(x^2yz + 4xz^2)$$

$$+ \bar{k}\cdot\frac{\partial}{\partial z}(x^2yz + 4xz^2)$$

$$= \bar{i}(yz(2x) + 4z^2) + \bar{j}(x^2z + 0) + \bar{k}(x^2y + 4x(2z))$$

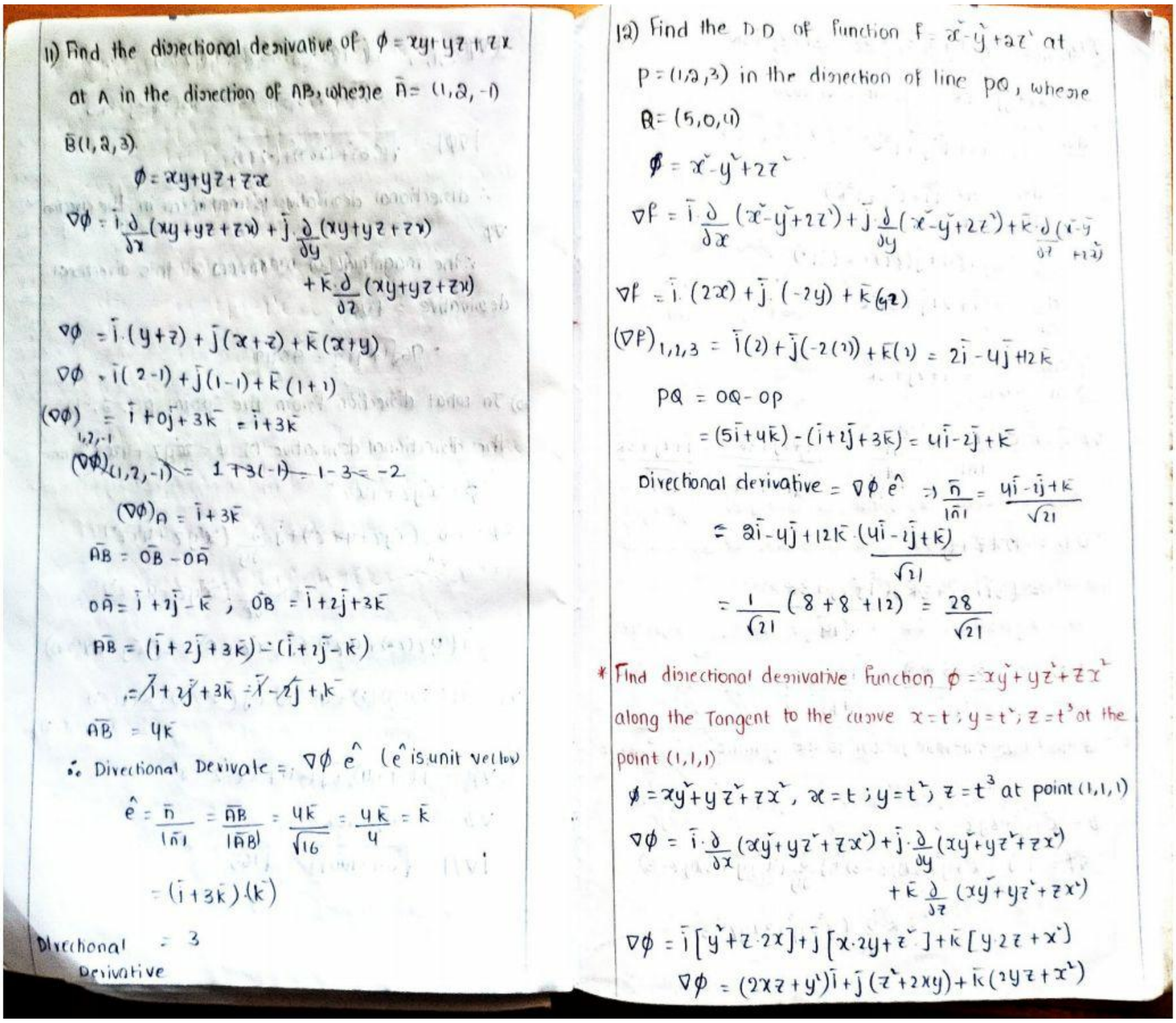
$$\nabla\phi_{(1,2,-1)} = \bar{i}((-2)(-1)(2) + 4(-1)^2] + \bar{j}(1(-1)) + \bar{k}(1(-2)$$

$$+ 4(1)(-2))$$

$$\nabla\phi = \bar{i}(4+4) + \bar{j}(-1) + \bar{k}(-2-8)$$

$$\nabla\phi = 8\bar{i} - \bar{j} - 10\bar{k}$$

$$|\nabla\phi| = \sqrt{64+100+1} = \sqrt{165}$$



11) Find the directional derivative of $\phi = xy + yz + zx$ at A in the direction of AB, where $\bar{A} = (1,2,-1)$ $\bar{B}(1,2,3)$.

$$\phi = xy + yz + zx$$

$$\nabla\phi = \bar{i}\cdot\frac{\partial}{\partial x}(xy+yz+zx) + \bar{j}\cdot\frac{\partial}{\partial y}(xy+yz+zx) + \bar{k}\cdot\frac{\partial}{\partial z}(xy+yz+zx)$$

$$\nabla\phi = \bar{i}\cdot(y+z) + \bar{j}(x+z) + \bar{k}(x+y)$$

$$\nabla\phi = \bar{i}(2-1) + \bar{j}(1-1) + \bar{k}(1+1)$$

$$(\nabla\phi)_{1,2,-1} = \bar{i} + 0\bar{j} + 3\bar{k} = \bar{i} + 3\bar{k}$$

$$(\nabla\phi)_{(1,2,-1)} = 1 + 3(-1) = 1 - 3 = -2$$

$$(\nabla\phi)_A = \bar{i} + 3\bar{k}$$

$$\bar{AB} = \bar{OB} - \bar{OA}$$

$$\bar{OA} = \bar{i} + 2\bar{j} - \bar{k}\ ;\quad \bar{OB} = \bar{i} + 2\bar{j} + 3\bar{k}$$

$$\bar{AB} = (\bar{i} + 2\bar{j} + 3\bar{k}) - (\bar{i} + 2\bar{j} - \bar{k})$$

$$= \cancel{\bar{i}} + \cancel{2\bar{j}} + 3\bar{k} - \cancel{\bar{i}} - \cancel{2\bar{j}} + \bar{k}$$

$$\bar{AB} = 4\bar{k}$$

$\therefore$ Directional Derivate $= \nabla\phi\ \hat{e}$ ($\hat{e}$ is unit vector)

$$\hat{e} = \frac{\bar{n}}{|\bar{n}|} = \frac{\bar{AB}}{|\bar{AB}|} = \frac{4\bar{k}}{\sqrt{16}} = \frac{4\bar{k}}{4} = \bar{k}$$

$$= (\bar{i} + 3\bar{k})\cdot(\bar{k})$$

Directional Derivative $= 3$

12) Find the D.D of function $f = x^2 - y^2 + 2z^2$ at $P = (1,2,3)$ in the direction of line PQ, where $Q = (5,0,4)$

$$\phi = x^2 - y^2 + 2z^2$$

$$\nabla f = \bar{i}\,\frac{\partial}{\partial x}(x^2 - y^2 + 2z^2) + \bar{j}\,\frac{\partial}{\partial y}(x^2 - y^2 + 2z^2) + \bar{k}\cdot\frac{\partial}{\partial z}(x^2 - y^2 + 2z^2)$$

$$\nabla f = \bar{i}\cdot(2x) + \bar{j}\cdot(-2y) + \bar{k}(4z)$$

$$(\nabla f)_{1,2,3} = \bar{i}(2) + \bar{j}(-2(2)) + \bar{k}(3) = 2\bar{i} - 4\bar{j} + 12\bar{k}$$

$$PQ = OQ - OP$$

$$= (5\bar{i} + 4\bar{k}) - (\bar{i} + 2\bar{j} + 3\bar{k}) = 4\bar{i} - 2\bar{j} + \bar{k}$$

Directional derivative $= \nabla\phi\ \hat{e} \Rightarrow \frac{\bar{n}}{|\bar{n}|} = \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}$

$$= 2\bar{i} - 4\bar{j} + 12\bar{k}\cdot\frac{(4\bar{i} - 2\bar{j} + \bar{k})}{\sqrt{21}}$$

$$= \frac{1}{\sqrt{21}}(8 + 8 + 12) = \frac{28}{\sqrt{21}}$$

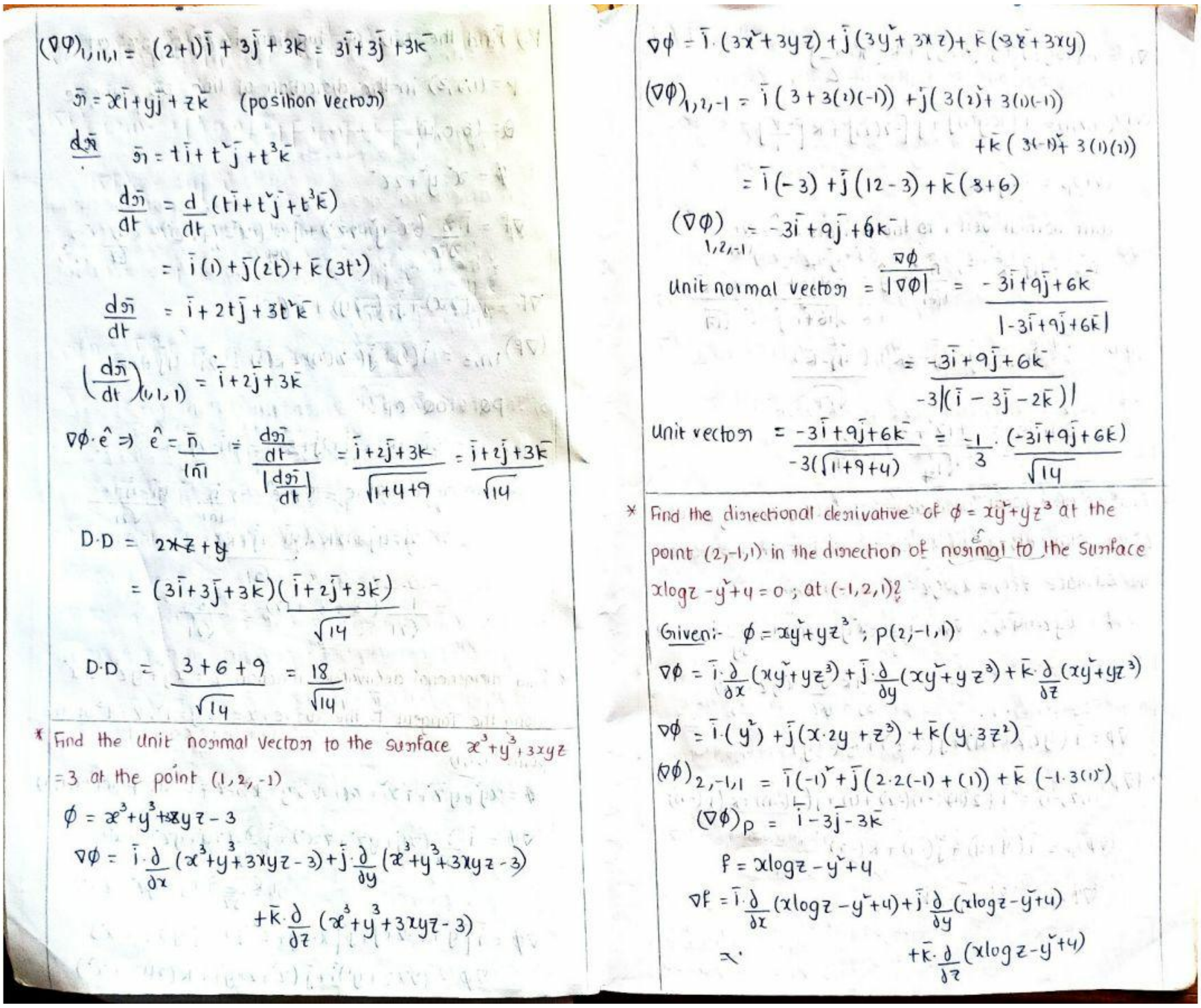
* Find directional derivative function $\phi = xy^2 + yz^2 + zx^2$ along the Tangent to the curve $x = t\ ;\ y = t^2\ ;\ z = t^3$ at the point $(1,1,1)$

$\phi = xy^2 + yz^2 + zx^2$, $x = t\ ;\ y = t^2\ ;\ z = t^3$ at point $(1,1,1)$

$$\nabla\phi = \bar{i}\cdot\frac{\partial}{\partial x}(xy^2 + yz^2 + zx^2) + \bar{j}\cdot\frac{\partial}{\partial y}(xy^2 + yz^2 + zx^2) + \bar{k}\,\frac{\partial}{\partial z}(xy^2 + yz^2 + zx^2)$$

$$\nabla\phi = \bar{i}\,[y^2 + z\cdot 2x] + \bar{j}\,[x\cdot 2y + z^2] + \bar{k}\,[y\cdot 2z + x^2]$$

$$\nabla\phi = (2xz + y^2)\bar{i} + \bar{j}(z^2 + 2xy) + \bar{k}(2yz + x^2)$$



$$(\nabla\phi)_{1,1,1} = (2+1)\bar{i} + 3\bar{j} + 3\bar{k} = 3\bar{i}+3\bar{j}+3\bar{k}$$

$\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ (position vector)

$\frac{d\bar{r}}{\;}$ $\bar{r} = t\bar{i}+t^2\bar{j}+t^3\bar{k}$

$$\frac{d\bar{r}}{dt} = \frac{d}{dt}(t\bar{i}+t^2\bar{j}+t^3\bar{k})$$

$$= \bar{i}(1)+\bar{j}(2t)+\bar{k}(3t^2)$$

$$\frac{d\bar{r}}{dt} = \bar{i}+2t\bar{j}+3t^2\bar{k}$$

$$\left(\frac{d\bar{r}}{dt}\right)_{(1,1,1)} = \bar{i}+2\bar{j}+3\bar{k}$$

$$\nabla\phi\cdot\hat{e} \Rightarrow \hat{e} = \frac{\bar{n}}{|\bar{n}|} = \frac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{1+4+9}} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{14}}$$

$$D\cdot D = 2xz+y$$

$$= (3\bar{i}+3\bar{j}+3\bar{k})\frac{(\bar{i}+2\bar{j}+3\bar{k})}{\sqrt{14}}$$

$$D\cdot D = \frac{3+6+9}{\sqrt{14}} = \frac{18}{\sqrt{14}}$$

* Find the unit normal vector to the surface $x^3+y^3+3xyz = 3$ at the point $(1,2,-1)$

$$\phi = x^3+y^3+3xyz-3$$

$$\nabla\phi = \bar{i}\cdot\frac{\partial}{\partial x}(x^3+y^3+3xyz-3)+\bar{j}\cdot\frac{\partial}{\partial y}(x^3+y^3+3xyz-3)$$
$$+\bar{k}\cdot\frac{\partial}{\partial z}(x^3+y^3+3xyz-3)$$

$$\nabla\phi = \bar{i}\cdot(3x^2+3yz)+\bar{j}(3y^2+3xz)+\bar{k}(3z^2+3xy)$$

$$(\nabla\phi)_{1,2,-1} = \bar{i}(3+3(2)(-1))+\bar{j}(3(2)^2+3(1)(-1))$$
$$+\bar{k}(3(-1)^2+3(1)(2))$$

$$= \bar{i}(-3)+\bar{j}(12-3)+\bar{k}(3+6)$$

$$(\nabla\phi)_{1,2,-1} = -3\bar{i}+9\bar{j}+6\bar{k}$$

$$\text{Unit normal vector} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-3\bar{i}+9\bar{j}+6\bar{k}}{|-3\bar{i}+9\bar{j}+6\bar{k}|}$$

$$= \frac{-3\bar{i}+9\bar{j}+6\bar{k}}{-3|(\bar{i}-3\bar{j}-2\bar{k})|}$$

$$\text{Unit vector} = \frac{-3\bar{i}+9\bar{j}+6\bar{k}}{-3(\sqrt{1+9+4})} = \frac{-1}{3}\cdot\frac{(-3\bar{i}+9\bar{j}+6\bar{k})}{\sqrt{14}}$$

* Find the directional derivative of $\phi = xy^2+yz^3$ at the point $(2,-1,1)$ in the direction of normal to the surface $x\log z - y^2+4 = 0$; at $(-1,2,1)$?

Given:- $\phi = xy^2+yz^3$; $P(2,-1,1)$

$$\nabla\phi = \bar{i}\cdot\frac{\partial}{\partial x}(xy^2+yz^3)+\bar{j}\cdot\frac{\partial}{\partial y}(xy^2+yz^3)+\bar{k}\cdot\frac{\partial}{\partial z}(xy^2+yz^3)$$

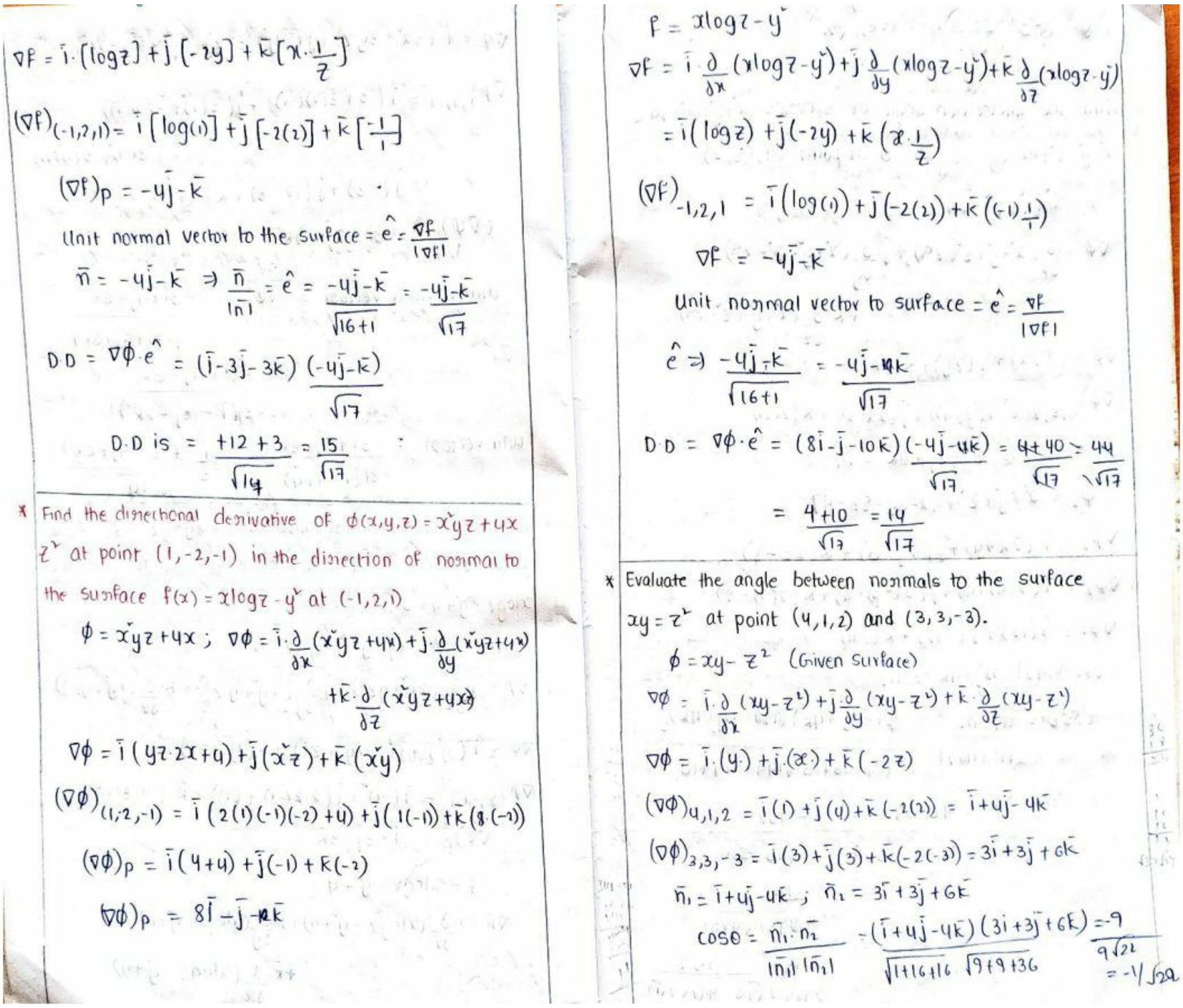
$$\nabla\phi = \bar{i}\cdot(y^2)+\bar{j}(x\cdot 2y+z^3)+\bar{k}(y\cdot 3z^2)$$

$$(\nabla\phi)_{2,-1,1} = \bar{i}(-1)^2+\bar{j}(2\cdot 2(-1)+(1))+\bar{k}(-1\cdot 3(1)^2)$$

$$(\nabla\phi)_P = \bar{i}-3\bar{j}-3\bar{k}$$

$$f = x\log z - y^2+4$$

$$\nabla f = \bar{i}\cdot\frac{\partial}{\partial x}(x\log z-y^2+4)+\bar{j}\cdot\frac{\partial}{\partial y}(x\log z-y^2+4)$$
$$+\bar{k}\cdot\frac{\partial}{\partial z}(x\log z-y^2+4)$$



$\nabla f = \bar{i}\cdot[\log z] + \bar{j}\,[-2y] + \bar{k}\,[x\cdot\frac{1}{z}]$

$(\nabla f)_{(-1,2,1)} = \bar{i}\,[\log(1)] + \bar{j}\,[-2(2)] + \bar{k}\,[\frac{-1}{1}]$

$(\nabla f)_P = -4\bar{j} - \bar{k}$

Unit normal vector to the surface $= \hat{e} = \frac{\nabla f}{|\nabla f|}$

$\bar{n} = -4\bar{j} - \bar{k} \Rightarrow \frac{\bar{n}}{|\bar{n}|} = \hat{e} = \frac{-4\bar{j}-\bar{k}}{\sqrt{16+1}} = \frac{-4\bar{j}-\bar{k}}{\sqrt{17}}$

$D\cdot D = \nabla\phi\cdot\hat{e} = (\bar{i} - 3\bar{j} - 3\bar{k})\,\frac{(-4\bar{j}-\bar{k})}{\sqrt{17}}$

$D\cdot D \text{ is } = \frac{+12+3}{\sqrt{14}} = \frac{15}{\sqrt{17}}$

* Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at point $(1,-2,-1)$ in the direction of normal to the surface $f(x) = x\log z - y^2$ at $(-1,2,1)$

$\phi = x^2yz + 4x$; $\nabla\phi = \bar{i}\cdot\frac{\partial}{\partial x}(x^2yz+4x) + \bar{j}\cdot\frac{\partial}{\partial y}(x^2yz+4x)$

$+\bar{k}\cdot\frac{\partial}{\partial z}(x^2yz+4x)$

$\nabla\phi = \bar{i}\,(yz\cdot 2x+4) + \bar{j}\,(x^2z) + \bar{k}\,(x^2y)$

$(\nabla\phi)_{(1,-2,-1)} = \bar{i}\,(2(1)(-1)(-2)+4) + \bar{j}\,(1(-1)) + \bar{k}\,(8\cdot(-2))$

$(\nabla\phi)_P = \bar{i}\,(4+4) + \bar{j}(-1) + \bar{k}(-2)$

$\nabla\phi)_P = 8\bar{i} - \bar{j} - 10\bar{k}$

$f = x\log z - y^2$

$\nabla f = \bar{i}\,\frac{\partial}{\partial x}(x\log z - y^2) + \bar{j}\,\frac{\partial}{\partial y}(x\log z - y^2) + \bar{k}\,\frac{\partial}{\partial z}(x\log z - y^2)$

$= \bar{i}(\log z) + \bar{j}(-2y) + \bar{k}\,(x\cdot\frac{1}{z})$

$(\nabla f)_{-1,2,1} = \bar{i}\,(\log(1)) + \bar{j}\,(-2(2)) + \bar{k}\,((-1)\frac{1}{1})$

$\nabla f = -4\bar{j} - \bar{k}$

Unit normal vector to surface $= \hat{e} = \frac{\nabla f}{|\nabla f|}$

$\hat{e} \Rightarrow \frac{-4\bar{j}-\bar{k}}{\sqrt{16+1}} = \frac{-4\bar{j}-\bar{k}}{\sqrt{17}}$

$D\cdot D = \nabla\phi\cdot\hat{e} = (8\bar{i} - \bar{j} - 10\bar{k})\frac{(-4\bar{j}-\bar{k})}{\sqrt{17}} = \frac{4+40}{\sqrt{17}} = \frac{44}{\sqrt{17}}$

$= \frac{4+10}{\sqrt{17}} = \frac{14}{\sqrt{17}}$

* Evaluate the angle between normals to the surface $xy = z^2$ at point $(4,1,2)$ and $(3,3,-3)$.

$\phi = xy - z^2$ (Given surface)

$\nabla\phi = \bar{i}\cdot\frac{\partial}{\partial x}(xy - z^2) + \bar{j}\,\frac{\partial}{\partial y}(xy - z^2) + \bar{k}\cdot\frac{\partial}{\partial z}(xy - z^2)$

$\nabla\phi = \bar{i}\cdot(y) + \bar{j}\cdot(x) + \bar{k}\,(-2z)$

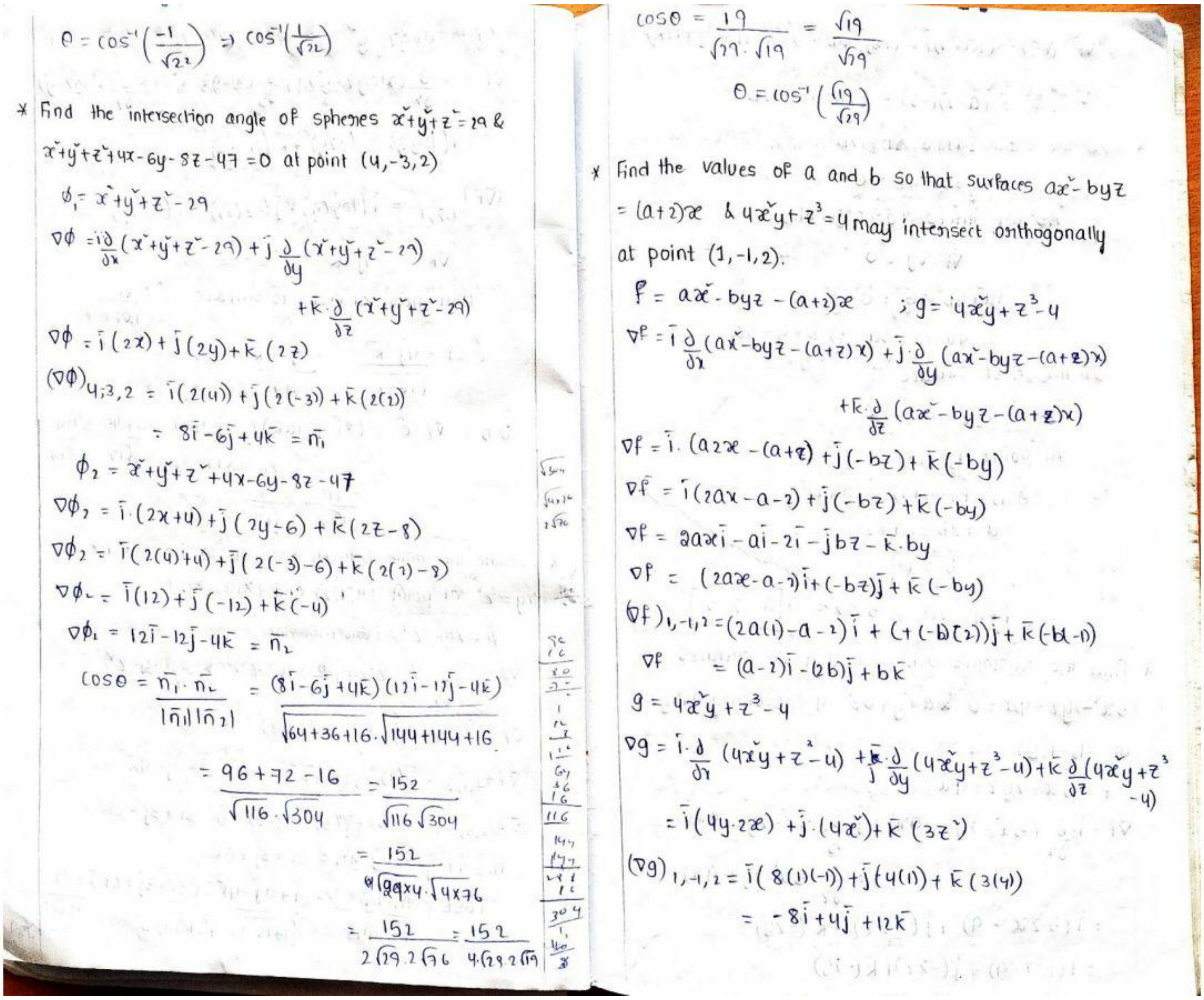
$(\nabla\phi)_{4,1,2} = \bar{i}(1) + \bar{j}\,(4) + \bar{k}\,(-2(2)) = \bar{i} + 4\bar{j} - 4\bar{k}$

$(\nabla\phi)_{3,3,-3} = \bar{i}\,(3) + \bar{j}\,(3) + \bar{k}(-2(-3)) = 3\bar{i} + 3\bar{j} + 6\bar{k}$

$\bar{n}_1 = \bar{i} + 4\bar{j} - 4\bar{k}$; $\bar{n}_2 = 3\bar{i} + 3\bar{j} + 6\bar{k}$

$\cos\theta = \frac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1|\,|\bar{n}_2|} = \frac{(\bar{i}+4\bar{j}-4\bar{k})(3\bar{i}+3\bar{j}+6\bar{k})}{\sqrt{1+16+16}\,\sqrt{9+9+36}} = \frac{-9}{9\sqrt{22}}$

$= -1/\sqrt{22}$



$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{22}}\right) \Rightarrow \cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$

* Find the intersection angle of spheres $x^2+y^2+z^2=29$ & $x^2+y^2+z^2+4x-6y-8z-47=0$ at point $(4,-3,2)$

$\phi_1 = x^2+y^2+z^2-29$

$\nabla\phi = \bar{i}\frac{\partial}{\partial x}(x^2+y^2+z^2-29) + \bar{j}\frac{\partial}{\partial y}(x^2+y^2+z^2-29) + \bar{k}\cdot\frac{\partial}{\partial z}(x^2+y^2+z^2-29)$

$\nabla\phi = \bar{i}(2x)+\bar{j}(2y)+\bar{k}(2z)$

$(\nabla\phi)_{4;3,2} = \bar{i}(2(4))+\bar{j}(2(-3))+\bar{k}(2(2))$

$= 8\bar{i}-6\bar{j}+4\bar{k} = \bar{n}_1$

$\phi_2 = x^2+y^2+z^2+4x-6y-8z-47$

$\nabla\phi_2 = \bar{i}\cdot(2x+4)+\bar{j}(2y-6)+\bar{k}(2z-8)$

$\nabla\phi_2 = \bar{i}(2(4)+4)+\bar{j}(2(-3)-6)+\bar{k}(2(2)-8)$

$\nabla\phi_2 = \bar{i}(12)+\bar{j}(-12)+\bar{k}(-4)$

$\nabla\phi_2 = 12\bar{i}-12\bar{j}-4\bar{k} = \bar{n}_2$

$\cos\theta = \frac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1||\bar{n}_2|} = \frac{(8\bar{i}-6\bar{j}+4\bar{k})(12\bar{i}-12\bar{j}-4\bar{k})}{\sqrt{64+36+16}\cdot\sqrt{144+144+16}}$

$= \frac{96+72-16}{\sqrt{116}\cdot\sqrt{304}} = \frac{152}{\sqrt{116}\sqrt{304}}$

$= \frac{152}{\sqrt{29\times4}\cdot\sqrt{4\times76}}$

$= \frac{152}{2\sqrt{29}.2\sqrt{76}} = \frac{152}{4\cdot\sqrt{29}.2\sqrt{19}}$

$\sqrt{304}$ $\sqrt{4\times76}$ $2\sqrt{76}$

$\cos\theta = \frac{19}{\sqrt{29}\cdot\sqrt{19}} = \frac{\sqrt{19}}{\sqrt{29}}$

$\theta = \cos^{-1}\left(\frac{\sqrt{19}}{\sqrt{29}}\right)$

* Find the values of a and b so that surfaces $ax^2-byz=(a+2)x$ & $4x^2y+z^3=4$ may intersect orthogonally at point $(1,-1,2)$.

$f = ax^2-byz-(a+2)x \quad ; g = 4x^2y+z^3-4$

$\nabla f = \bar{i}\frac{\partial}{\partial x}(ax^2-byz-(a+2)x) + \bar{j}\cdot\frac{\partial}{\partial y}(ax^2-byz-(a+2)x) + \bar{k}\cdot\frac{\partial}{\partial z}(ax^2-byz-(a+2)x)$

$\nabla f = \bar{i}\cdot(a2x-(a+2)) + \bar{j}(-bz)+\bar{k}(-by)$

$\nabla f = \bar{i}(2ax-a-2)+\bar{j}(-bz)+\bar{k}(-by)$

$\nabla f = 2ax\bar{i}-a\bar{i}-2\bar{i}-\bar{j}bz-\bar{k}\cdot by$

$\nabla f = (2ax-a-2)\bar{i}+(-bz)\bar{j}+\bar{k}(-by)$

$(\nabla f)_{1,-1,2} = (2a(1)-a-2)\bar{i} + (+(-b)(2))\bar{j}+\bar{k}(-b(-1))$

$\nabla f = (a-2)\bar{i}-(2b)\bar{j}+b\bar{k}$

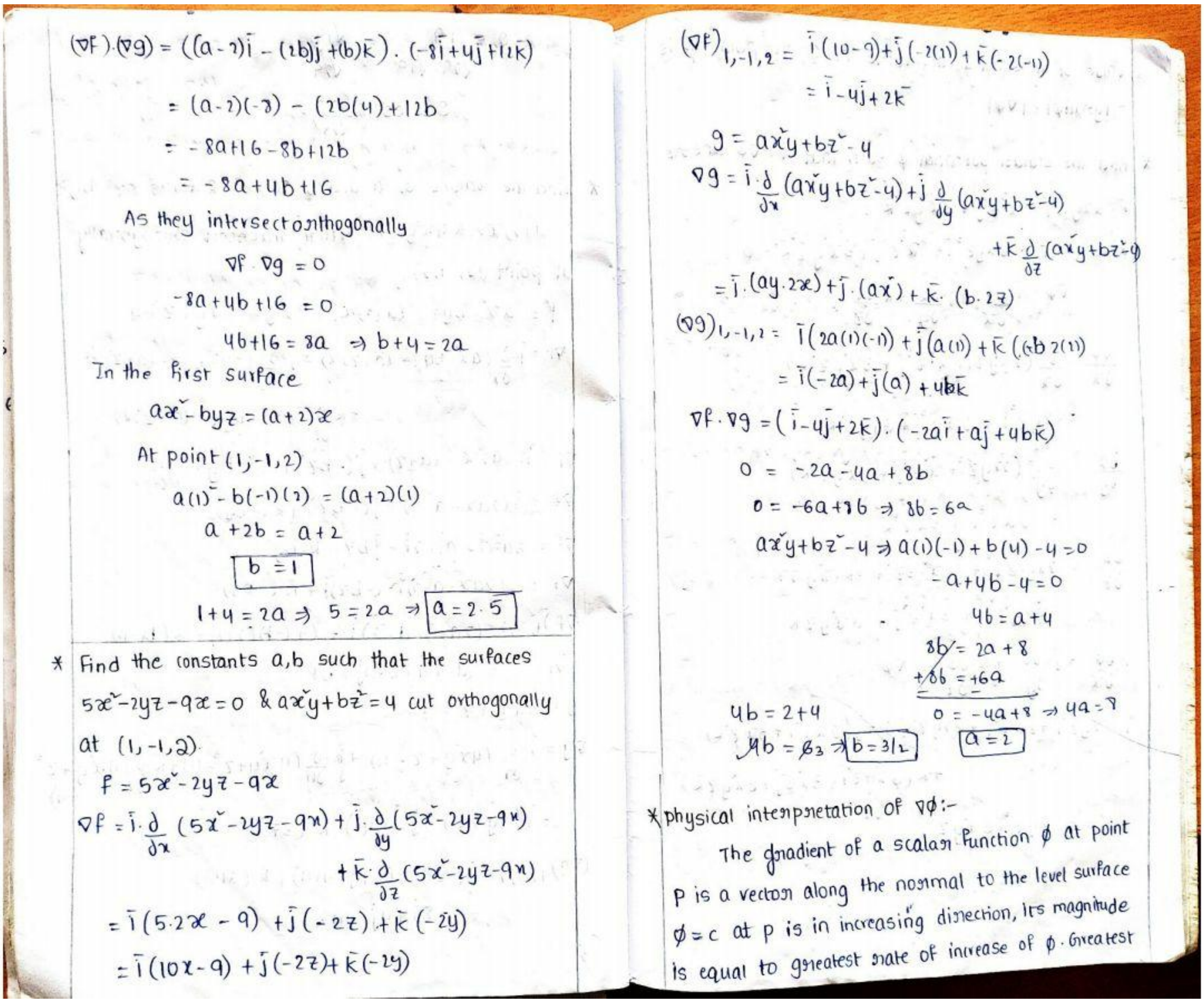
$g = 4x^2y+z^3-4$

$\nabla g = \bar{i}\cdot\frac{\partial}{\partial x}(4x^2y+z^2-4) + \bar{j}\frac{\partial}{\partial y}(4x^2y+z^3-4)+\bar{k}\frac{\partial}{\partial z}(4x^2y+z^3-4)$

$= \bar{i}(4y\cdot2x)+\bar{j}\cdot(4x^2)+\bar{k}(3z^2)$

$(\nabla g)_{1,-1,2} = \bar{i}(8(1)(-1))+\bar{j}(4(1))+\bar{k}(3(4))$

$= -8\bar{i}+4\bar{j}+12\bar{k}$



$(\nabla f)\cdot(\nabla g) = ((a-2)\bar{i} - (2b)\bar{j} + (b)\bar{k}) \cdot (-8\bar{i} + 4\bar{j} + 12\bar{k})$

$= (a-2)(-8) - (2b(4) + 12b$

$= -8a + 16 - 8b + 12b$

$= -8a + 4b + 16$

As they intersect orthogonally

$\nabla f \cdot \nabla g = 0$

$-8a + 4b + 16 = 0$

$4b + 16 = 8a \Rightarrow b + 4 = 2a$

In the first surface

$ax^2 - byz = (a+2)x$

At point $(1,-1,2)$

$a(1)^2 - b(-1)(2) = (a+2)(1)$

$a + 2b = a + 2$

$\boxed{b = 1}$

$1 + 4 = 2a \Rightarrow 5 = 2a \Rightarrow \boxed{a = 2.5}$

* Find the constants a, b such that the surfaces $5x^2 - 2yz - 9x = 0$ & $ax^2y + bz^2 = 4$ cut orthogonally at $(1,-1,2)$.

$f = 5x^2 - 2yz - 9x$

$\nabla f = \bar{i}\cdot\frac{\partial}{\partial x}(5x^2 - 2yz - 9x) + \bar{j}\cdot\frac{\partial}{\partial y}(5x^2 - 2yz - 9x) + \bar{k}\cdot\frac{\partial}{\partial z}(5x^2 - 2yz - 9x)$

$= \bar{i}(5.2x - 9) + \bar{j}(-2z) + \bar{k}(-2y)$

$= \bar{i}(10x - 9) + \bar{j}(-2z) + \bar{k}(-2y)$

$(\nabla f)_{1,-1,2} = \bar{i}(10-9) + \bar{j}(-2(2)) + \bar{k}(-2(-1))$

$= \bar{i} - 4\bar{j} + 2\bar{k}$

$g = ax^2y + bz^2 - 4$

$\nabla g = \bar{i}\cdot\frac{\partial}{\partial x}(ax^2y + bz^2 - 4) + \bar{j}\,\frac{\partial}{\partial y}(ax^2y + bz^2 - 4) + \bar{k}\,\frac{\partial}{\partial z}(ax^2y + bz^2 - 4)$

$= \bar{i}\cdot(ay.2x) + \bar{j}\cdot(ax^2) + \bar{k}\cdot(b.2z)$

$(\nabla g)_{1,-1,2} = \bar{i}(2a(1)(-1)) + \bar{j}(a(1)) + \bar{k}((b\,2(2))$

$= \bar{i}(-2a) + \bar{j}(a) + 4b\bar{k}$

$\nabla f \cdot \nabla g = (\bar{i} - 4\bar{j} + 2\bar{k})\cdot(-2a\bar{i} + a\bar{j} + 4b\bar{k})$

$0 = -2a - 4a + 8b$

$0 = -6a + 8b \Rightarrow 8b = 6a$

$ax^2y + bz^2 - 4 \Rightarrow a(1)(-1) + b(4) - 4 = 0$

$-a + 4b - 4 = 0$

$4b = a + 4$

$\cancel{8b} = 2a + 8$

$+\cancel{8b} = +6a$

$0 = -4a + 8 \Rightarrow 4a = 8$

$\boxed{a = 2}$

$4b = 2 + 4$

$4b = \cancel{6}3 \Rightarrow \boxed{b = 3/2}$

* physical interpretation of $\nabla\phi$:-

The gradient of a scalar function ϕ at point P is a vector along the normal to the level surface $\phi = c$ at P is in increasing direction, Its magnitude is equal to greatest rate of increase of ϕ. Greatest value of directional derivative of ϕ at point p is

$= |\text{grad}\phi| = |\nabla\phi|$

* Find the scalar potential ϕ such that $\bar{F} = \nabla\phi$ where

$\bar{F} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k}$

Given:- $\bar{F} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k} - \phi$

$$\nabla\phi = \bar{i}\cdot\frac{\partial\phi}{\partial x} + \bar{j}\cdot\frac{\partial\phi}{\partial y} + \bar{k}\cdot\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}(2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k})$$

$$= 2yz^3\bar{i} + 2xz^3\bar{j} + 6xyz^2\bar{k}$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}(2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k})$$

$$= 2xz^3\bar{i} + 3x^2z^2\bar{k}$$

$$\frac{\partial\phi}{\partial z} = 2xy\cdot 3z^2\bar{i} + \bar{j}\cdot x^2\cdot 3z^2 + 3\bar{k}\cdot x^2y\cdot 2z$$

$$= 6xyz^2\bar{i} + 3x^2z^2\bar{j} + 6x^2yz\bar{k}$$

$$\bar{i}\cdot\frac{\partial\phi}{\partial x} + \bar{j}\cdot\frac{\partial\phi}{\partial y} + \bar{k}\cdot\frac{\partial\phi}{\partial z} = \nabla\phi$$

$$\bar{i}\cdot(2yz^3\bar{i} + 2xz^3\bar{j} + 6xyz^2\bar{k}) + \bar{j}(2xz^3\bar{i} + 3x^2z^2\bar{k}) + \bar{k}(6xyz^2\bar{i} + 3x^2z^2\bar{j} + 6x^2yz\bar{k})$$

$$\nabla\phi = 2yz^3 + 6x^2yz$$

$$2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k} = \bar{i}\cdot\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\cdot\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xyz^3$$

Integrate On both sides

$$\int\frac{\partial\phi}{\partial x} = \int 2xyz^3\,dx$$

$$\phi = 2\frac{x^2}{2}yz^3 + C$$

$$\phi = x^2yz^3 + C$$

$$\frac{\partial\phi}{\partial y} = x^2z^3$$

Integrate On Both sides

$$\int\frac{\partial\phi}{\partial y}\,dy = \int x^2z^3\,dy$$

$$= x^2z^3y + C$$

$$\frac{\partial\phi}{\partial z} = 3x^2yz^2$$

$$\int\frac{\partial\phi}{\partial z}\,dz = 3x^2y\int z^2\,dz = \frac{3x^2yz^3}{3} = x^2yz^3 + C$$

$$\boxed{\phi = x^2yz^3 + C}$$

Curl of a vector:- If $\bar{F}$ is any differentiable vector point function then the vector function is defined by $\bar{i}\cdot\frac{\partial\bar{F}}{\partial x} + \bar{j}\frac{\partial\bar{F}}{\partial y} + \bar{k}\cdot\frac{\partial F}{\partial z}$ is called a curl of vector point function. And it is denoted by <u>curl $\bar{F}$ or $\nabla\times\bar{F}$</u>

If $\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then $\text{curl}\,\bar{F} = \nabla\times\bar{F} = \left(\bar{i}\cdot\frac{\partial}{\partial x} + \bar{j}\cdot\frac{\partial}{\partial y} + \bar{k}\cdot\frac{\partial}{\partial z}\right)$ $\times(f_1\bar{i} + f_2\bar{j} + f_3\bar{k})$; $\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$

Irrotational Vector:- A vector point function $\bar{F}$ is said to be irrotationaly if <u>curl $\bar{F} = 0$</u>

Value of directional derivative of ϕ at point p is
$= |\text{grad}\,\phi| = |\nabla\phi|$

* Find the scalar potential ϕ such that $\bar{F} = \nabla\phi$ where
$\bar{F} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k}$

Given:- $\bar{F} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k} = \phi$

$$\nabla\phi = \bar{i}\cdot\frac{\partial\phi}{\partial x} + \bar{j}\cdot\frac{\partial\phi}{\partial y} + \bar{k}\cdot\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}(2xyz^3\bar{i} + x^2\cdot z^3\bar{j} + 3x^2yz^2\bar{k})$$
$$= 2yz^3\bar{i} + 2xz^3\bar{j} + 6xyz^2\bar{k}$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}(2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k})$$
$$= 2xz^3\bar{i} + 3x^2z^2\bar{k}$$

$$\frac{\partial\phi}{\partial z} = 2xy\cdot 3z^2\bar{i} + \bar{j}\cdot x^2\cdot 3z^2 + 3\bar{k}\cdot x^2y\cdot 2z$$
$$= 6xyz^2\bar{i} + 3x^2z^2\bar{j} + 6x^2yz\bar{k}$$

$$\bar{i}\cdot\frac{\partial\phi}{\partial x} + \bar{j}\cdot\frac{\partial\phi}{\partial y} + \bar{k}\cdot\frac{\partial\phi}{\partial z} = \nabla\phi$$

$$\bar{i}\cdot(2yz^3\bar{i} + 2xz^3\bar{j} + 6xyz^2\bar{k}) + \bar{j}(2xz^3\bar{i} + 3x^2z^2\bar{k})$$
$$+ \bar{k}(6xyz^2\bar{i} + 3x^2z^2\bar{j} + 6x^2yz\bar{k})$$

$$\nabla\phi = 2yz^3 + 6x^2yz$$

$$2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k} = \bar{i}\cdot\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xyz^3$$

Integrate on both sides

$$\int\frac{\partial\phi}{\partial x} = \int 2xyz^3\,dx$$
$$\phi = 2\frac{x^2}{2}yz^3 + C$$
$$\phi = x^2yz^3 + C$$

$$\frac{\partial\phi}{\partial y} = x^2z^3$$

Integrate On Both sides

$$\int\frac{\partial\phi}{\partial y}\,dy = \int x^2z^3\,dy$$
$$= x^2z^3y + C$$

$$\frac{\partial\phi}{\partial z} = 3x^2yz^2$$

$$\int\frac{\partial\phi}{\partial z}\,dz = 3x^2y\int z^2\,dz = \frac{3x^2yz^3}{3} = x^2yz^3 + c$$

$$\boxed{\phi = x^2yz^3 + C}$$

Curl of a vector:- If $\bar{F}$ is any differentiable vector point function then the vector function is defined by $\bar{i}\cdot\frac{\partial\bar{F}}{\partial x} + \bar{j}\frac{\partial\bar{F}}{\partial y} + \bar{k}\cdot\frac{\partial\bar{F}}{\partial z}$ is called a curl of vector point function. And it is denoted by <u>curl $\bar{F}$ or $\nabla\times\bar{F}$</u>

If $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ then $\text{curl}\,\bar{F} = \nabla\times\bar{F} = \left(\bar{i}\cdot\frac{\partial}{\partial x} + \bar{j}\cdot\frac{\partial}{\partial y} + \bar{k}\cdot\frac{\partial}{\partial z}\right)$
$\times(F_1\bar{i} + F_2\bar{j} + F_3\bar{k})$; $\text{curl}\,\bar{F} = \begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ F_1 & F_2 & F_3\end{vmatrix}$

Irrotational Vector:- A vector point function $\bar{F}$ is said to be irrotationaly if <u>curl $\bar{F} = 0$</u>

* If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ Find curl$\bar{f}$ at the point $(1,-1,1)$

$$\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$$

$$\text{curl}\bar{f} = \nabla\times\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right] - \bar{j}\left[\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right] + \bar{k}\left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]$$

$$= \bar{i}(-3z^2 - 2x^2y) - \bar{j}(0-0) + \bar{k}(2yz\cdot 2x - x\cdot 2y)$$

$$= \bar{i}(-2x^2y - 3z^2) + \bar{k}(-2xy + 4xyz)$$

$$(\text{curl}\bar{f})_{1,-1,1} = \bar{i}(-2 1.(-1) - 3(1)) + \bar{k}(-2(1)(-1) - 4(-1))$$

$$= \bar{i}(2-3) + \bar{k}(2 \mp 4)$$

$$= -\bar{i} \mp \underset{2}{6}\bar{k} = -\bar{i} - 2\bar{k}$$

* show that the vector $\bar{f} = e^{x+y+z}(\bar{i}+\bar{j}+\bar{k})$ is irrotational

$$\bar{f} = e^{x+y+z}(\bar{i}+\bar{j}+\bar{k})$$

$$\text{curl}\bar{f} = \nabla\times\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x+y+z} & e^{x+y+z} & e^{x+y+z} \end{vmatrix}$$

$$= e^{x+y+z} \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & 1 & 1 \end{vmatrix}$$

$$e^{x+y+z}\left[\bar{i}\left(\frac{\partial}{\partial y}(1) - \frac{\partial}{\partial z}(1)\right] - \bar{j}\left[\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial z}(1)\right] + \bar{k}\left[\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(1)\right]$$

$$= e^{x+y+z}[0] = 0$$

$\boxed{\text{curl}\bar{f} = 0}$ Hence given vector is irrotational.

* Find curl$\bar{f}$ where $\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$

let $\phi = x^3+y^3+z^3-3xyz$

$$\text{grad}(\phi) = \bar{i}\cdot\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz) + \bar{j}\cdot\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz) + \bar{k}\frac{\partial}{\partial z}(x^2+y^3+z^3-3xyz)$$

$$\bar{f} = \text{grad}\phi = \bar{i}(3x^2-3yz) + \bar{j}(3y^2-3xz) + \bar{k}(3z^2-3xy)$$

$$\therefore \bar{f} = \bar{i}(3x^2-3yz) + \bar{j}(3y^2-3xz) + \bar{k}(3z^2-3xy)$$

$$\text{curl}\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= \bar{i}(\underset{+3z}{-3z}) - \bar{j}(-3y+3y) + \bar{k}(-3z+3z)$$

curl$\bar{f} = 0 \Rightarrow$ curl(gradϕ) $= 0$

* Find constants a,b,c so that the vector $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + \bar{k}(4x+cy+2z)$ also find ϕ such that $\bar{A} = \nabla\phi$

$$\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + \bar{k}(4x+cy+2z)$$

$$\nabla\phi = \bar{i}\frac{\partial}{\partial x}$$

$$\text{curl}\,\bar{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right]$$

$$-\bar{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$$

$$+\bar{k}\cdot\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$

$$0 = \bar{i}(c+1) - \bar{j}(4-a) + \bar{k}(b-2)$$

$c=-1$; $a=+4$; $b=2$

$$\bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + \bar{k}(4x-y+2z)$$

$$A = \nabla\phi = \bar{i}\cdot\frac{\partial\phi}{\partial x} + \bar{j}\cdot\frac{\partial\phi}{\partial y} + \bar{k}\cdot\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = x+2y+4z \; ; \; \frac{\partial\phi}{\partial y} = 2x-3y-z \; ; \; \frac{\partial\phi}{\partial z} = 4x-y+2z$$

Integrate on Both sides

$$\int\frac{\partial\phi}{\partial x} = \int(x+2y+4z)\,dx = \frac{x^2}{2} + 2yx + 4zx = \phi$$

$$\int\frac{\partial\phi}{\partial y} = \int(2x-3y-z)\,dy = \cancel{2}\frac{x^2}{\cancel{2}} - \frac{3y^2}{2}\cancel{x} - zxy + c$$

$$\int\frac{\partial\phi}{\partial z} = \int(4x-y+2z)\,dz = 4\frac{xz}{2} - y\cancel{z} + 2\frac{z^2}{2}\cancel{x}$$

$$\phi = \frac{x^2}{2} + 2yx + 4zx + c + 2xy - \frac{3y^2}{2} - zy + c + 4xz - yz + z^2 + c$$

$$\phi = \frac{x^2}{2} + 4xy - 2yz + 8xz + z^2 - \frac{3y^2}{2} + c$$

show that Vector field $\bar{F} = 2xyz^2\bar{i} + (x^2z^2 + z\cos yz)\bar{j} + (2x^2yz + y\cos yz)\bar{k}$ is irrotational? find the scalar potential function?

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2+z\cos yz & 2x^2yz+y\cos yz \end{vmatrix}$$

$$\text{curl}\,\bar{F} = \bar{i}\left[\frac{\partial}{\partial y}(2x^2yz+y\cos yz) - \frac{\partial}{\partial z}(x^2z^2+z\cos yz)\right]$$

$$-\bar{j}\left[\frac{\partial}{\partial x}(2x^2yz+y\cos yz) - \frac{\partial}{\partial z}(2xyz^2)\right]$$

$$+\bar{k}\cdot\left[\frac{\partial}{\partial x}(x^2z^2+z\cos yz) - \frac{\partial}{\partial y}(2xyz^2)\right]$$

$$\text{curl}\,\bar{F} = \bar{i}\left[2x^2z(1) + \frac{\partial}{\partial y}(y\cos yz)\right] - \left[x^2\cdot(2z) + \frac{\partial}{\partial z}(z\cos yz)\right]$$

$$-\bar{j}\left[2yz\cdot 2x + \frac{\partial}{\partial x}(y\cos yz)\right] - 2xy(2z)\Big]$$

$$+\bar{k}\left[z^2\cdot 2x\right] - 2xz^2(1)\Big]$$

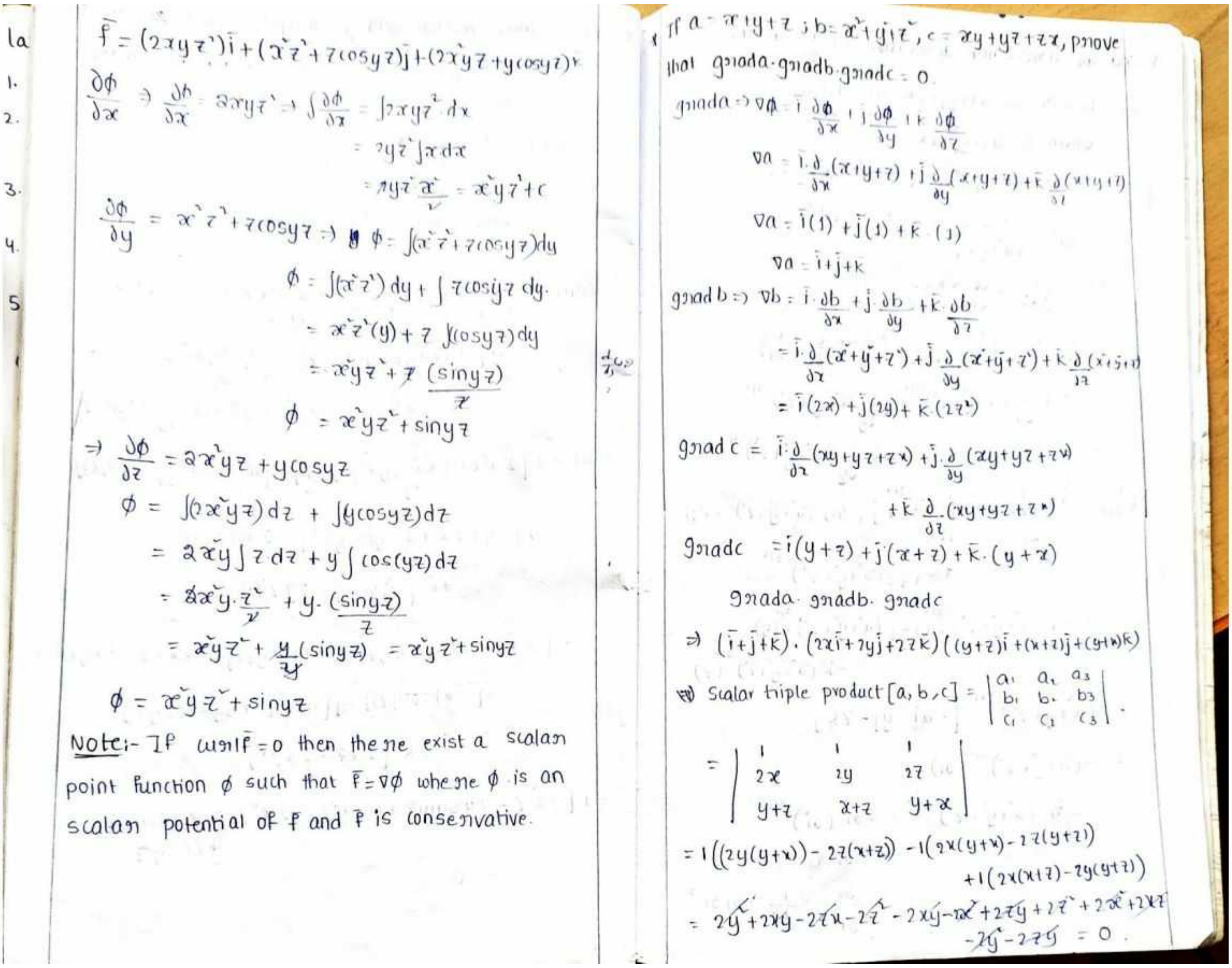
$$= \bar{i}\left[2x^2z + (y\cdot(-\sin yz)(z) + 1\cdot\cos yz)\right] - (2zx^2 + z\cdot(-\sin yz)(y)$$

$$+1\cdot\cos yz)) - \bar{j}\left[4xyz + 0\right] - 4xyz\Big]$$

$$+\bar{k}\left[2xz^2 - 2xz^2\right]$$

$$= \bar{i}\left[\cancel{2x^2z} - zy\sin yz + \cos yz - \cancel{2x^2z} + zy\sin yz - \cos yz\right]$$

$$= 0$$



la $\bar{f} = (2xyz^2)\bar{i} + (x^2z^2 + z\cos yz)\bar{j} + (2x^2yz + y\cos yz)k$

1\. $\frac{\partial\phi}{\partial x} \Rightarrow \frac{\partial\phi}{\partial x} = 2xyz^2 \Rightarrow \int\frac{\partial\phi}{\partial x} = \int 2xyz^2\, dx$

2\. $= 2yz^2\int x\, dx$

3\. $= 2yz^2\frac{x^2}{2} = x^2yz^2 + c$

$\frac{\partial\phi}{\partial y} = x^2z^2 + z\cos yz \Rightarrow \phi = \int(x^2z^2 + z\cos yz)dy$

4\. $\phi = \int(x^2z^2)\, dy + \int z\cos yz\, dy.$

5 $= x^2z^2(y) + z\int(\cos yz)\, dy$

$= x^2yz^2 + z\frac{(\sin yz)}{z}$

$\phi = x^2yz^2 + \sin yz$

$\Rightarrow \frac{\partial\phi}{\partial z} = 2x^2yz + y\cos yz$

$\phi = \int(2x^2yz)dz + \int(y\cos yz)dz$

$= 2x^2y\int z\, dz + y\int\cos(yz)\, dz$

$= 2x^2y.\frac{z^2}{2} + y.\frac{(\sin yz)}{z}$

$= x^2yz^2 + \frac{y}{y}(\sin yz) = x^2yz^2 + \sin yz$

$\phi = x^2yz^2 + \sin yz$

Note:- If curl $\bar{f} = 0$ then there exist a scalar point function ϕ such that $\bar{f} = \nabla\phi$ where ϕ is an scalar potential of f and f is conservative.

If $a = x + y + z$; $b = x^2 + y^2 + z^2$, $c = xy + yz + zx$, prove that grada·gradb·gradc = 0

grada $\Rightarrow \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$\nabla a = \bar{i}.\frac{\partial}{\partial x}(x+y+z) + \bar{j}\frac{\partial}{\partial y}(x+y+z) + \bar{k}\frac{\partial}{\partial z}(x+y+z)$

$\nabla a = \bar{i}(1) + \bar{j}(1) + \bar{k}\ (1)$

$\nabla a = \bar{i} + \bar{j} + k$

grad b $\Rightarrow \nabla b = \bar{i}.\frac{\partial b}{\partial x} + \bar{j}.\frac{\partial b}{\partial y} + \bar{k}.\frac{\partial b}{\partial z}$

$= \bar{i}.\frac{\partial}{\partial x}(x^2+y^2+z^2) + \bar{j}\frac{\partial}{\partial y}(x^2+y^2+z^2) + \bar{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$

$= \bar{i}(2x) + \bar{j}(2y) + \bar{k}(2z^2)$

grad c $= \bar{i}.\frac{\partial}{\partial x}(xy+yz+zx) + \bar{j}.\frac{\partial}{\partial y}(xy+yz+zx)$

$+ \bar{k}\frac{\partial}{\partial z}(xy+yz+zx)$

gradc $= \bar{i}(y+z) + \bar{j}(x+z) + \bar{k}.(y+x)$

grada. gradb. gradc

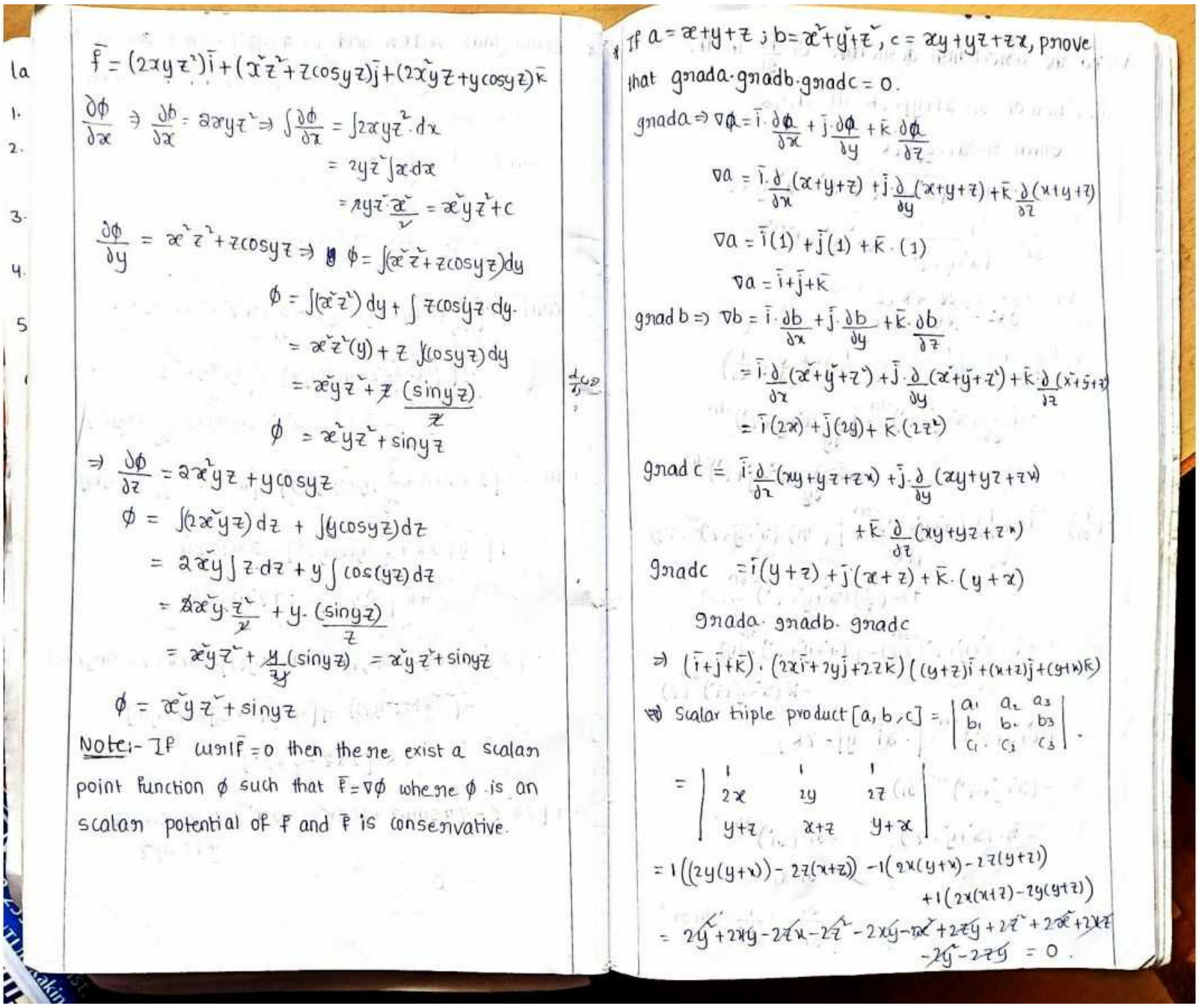
$\Rightarrow (\bar{i}+\bar{j}+\bar{k}).(2x\bar{i}+2y\bar{j}+2z\bar{k})((y+z)\bar{i}+(x+z)\bar{j}+(y+x)\bar{k})$

Scalar tiple product $[a, b, c] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$

$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & y+x \end{vmatrix}$

$= 1((2y(y+x)) - 2z(x+z)) - 1(2x(y+x) - 2z(y+z)) + 1(2x(x+z) - 2y(y+z))$

$= \cancel{2y^2} + 2xy - 2zx - 2z^2 - 2xy - 2x^2 + 2zy + 2z^2 + 2x^2 + 2xz - \cancel{2y^2} - 2zy = 0.$



$$\bar{f} = (2xyz^2)\bar{i} + (x^2z^2 + z\cos yz)\bar{j} + (2x^2yz + y\cos yz)\bar{k}$$

$$\frac{\partial\phi}{\partial x} \Rightarrow \frac{\partial\phi}{\partial x} = 2xyz^2 \Rightarrow \int\frac{\partial\phi}{\partial x} = \int 2xyz^2 \cdot dx$$

$$= 2yz^2\int x\,dx$$

$$= 2yz^2 \frac{x^2}{2} = x^2yz^2 + c$$

$$\frac{\partial\phi}{\partial y} = x^2z^2 + z\cos yz \Rightarrow \phi = \int(x^2z^2 + z\cos yz)dy$$

$$\phi = \int(x^2z^2)\,dy + \int z\cos yz \cdot dy.$$

$$= x^2z^2(y) + z\int(\cos yz)\,dy$$

$$= x^2yz^2 + \cancel{z}\,\frac{(\sin yz)}{\cancel{z}}$$

$$\phi = x^2yz^2 + \sin yz$$

$$\Rightarrow \frac{\partial\phi}{\partial z} = 2x^2yz + y\cos yz$$

$$\phi = \int(2x^2yz)\,dz + \int(y\cos yz)\,dz$$

$$= 2x^2y\int z \cdot dz + y\int\cos(yz)\,dz$$

$$= \cancel{2}x^2y \cdot \frac{z^2}{\cancel{2}} + y \cdot \frac{(\sin yz)}{z}$$

$$= x^2yz^2 + \frac{\cancel{y}}{\cancel{y}}(\sin yz) = x^2yz^2 + \sin yz$$

$$\phi = x^2yz^2 + \sin yz$$

Note:- If curl $\bar{F} = 0$ then there exist a scalar point function ϕ such that $\bar{F} = \nabla\phi$ where ϕ is an scalar potential of $\bar{F}$ and $\bar{F}$ is conservative.

If $a = x+y+z$; $b = x^2+y^2+z^2$, $c = xy+yz+zx$, prove that grada·gradb·gradc = 0.

$$\text{grad}\,a \Rightarrow \nabla\phi = \bar{i}\cdot\frac{\partial\phi}{\partial x} + \bar{j}\,\frac{\partial\phi}{\partial y} + \bar{k}\,\frac{\partial\phi}{\partial z}$$

$$\nabla a = \bar{i}\cdot\frac{\partial}{\partial x}(x+y+z) + \bar{j}\,\frac{\partial}{\partial y}(x+y+z) + \bar{k}\,\frac{\partial}{\partial z}(x+y+z)$$

$$\nabla a = \bar{i}(1) + \bar{j}(1) + \bar{k}\cdot(1)$$

$$\nabla a = \bar{i}+\bar{j}+\bar{k}$$

$$\text{grad}\,b \Rightarrow \nabla b = \bar{i}\cdot\frac{\partial b}{\partial x} + \bar{j}\cdot\frac{\partial b}{\partial y} + \bar{k}\cdot\frac{\partial b}{\partial z}$$

$$= \bar{i}\cdot\frac{\partial}{\partial x}(x^2+y^2+z^2) + \bar{j}\cdot\frac{\partial}{\partial y}(x^2+y^2+z^2) + \bar{k}\,\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= \bar{i}(2x) + \bar{j}(2y) + \bar{k}\cdot(2z^2)$$

$$\text{grad}\,c = \bar{i}\cdot\frac{\partial}{\partial x}(xy+yz+zx) + \bar{j}\cdot\frac{\partial}{\partial y}(xy+yz+zx)$$

$$+ \bar{k}\,\frac{\partial}{\partial z}(xy+yz+zx)$$

$$\text{grad}\,c = \bar{i}(y+z) + \bar{j}(x+z) + \bar{k}\cdot(y+x)$$

grada· gradb· gradc

$$\Rightarrow (\bar{i}+\bar{j}+\bar{k})\cdot(2x\bar{i}+2y\bar{j}+2z\bar{k})\left((y+z)\bar{i}+(x+z)\bar{j}+(y+x)\bar{k}\right)$$

Scalar triple product $[a, b, c] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & y+x \end{vmatrix}$$

$$= 1\left((2y(y+x)) - 2z(x+z)\right) - 1\left(2x(y+x) - 2z(y+z)\right)$$

$$+ 1\left(2x(x+z) - 2y(y+z)\right)$$

$$= \cancel{2y^2} + 2xy - \cancel{2zx} - \cancel{2z^2} - 2xy - \cancel{2x^2} + \cancel{2zy} + \cancel{2z^2} + \cancel{2x^2} + \cancel{2xz}$$

$$- \cancel{2y^2} - \cancel{2zy} = 0.$$

* Find the directional derivative of $\frac{1}{r}$ in the

1a direction of $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ at $(1,1,2)$

1\. Given $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

2\. $\bar{r} = |\bar{r}| = \sqrt{x^2+y^2+z^2}$; $r^2 = x^2+y^2+z^2$

3\. $\frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-1/2}$

4\. $\nabla\phi = \bar{i}\cdot\frac{\partial\phi}{\partial x} + \bar{j}\cdot\frac{\partial\phi}{\partial y} + \bar{k}\cdot\frac{\partial\phi}{\partial z}$

5 $\nabla\left(\frac{1}{r}\right) = \bar{i}\cdot\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \bar{j}\cdot\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \bar{k}\cdot\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$$= \bar{i}\cdot\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + \bar{j}\cdot\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} + \bar{k}\cdot\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}$$

$$\nabla\left(\frac{1}{r}\right) = \bar{i}\cdot\left(\frac{-1}{2}\right)\cdot(x^2+y^2+z^2)^{-3/2}(2x) + \bar{j}\cdot(-1/2)\cdot(x^2+y^2+z^2)^{-3/2}\cdot(2y) + \bar{k}\left(\frac{-1}{2}\right)(x^2+y^2+z^2)^{-3/2}\cdot(2z)$$

$$= -\bar{i}\,(x^2+y^2+z^2)^{-3/2}(x) - \bar{j}\cdot(x^2+y^2+z^2)^{-3/2}\cdot(y) - \bar{k}\,(x^2+y^2+z^2)^{-3/2}(z)$$

$$= (x^2+y^2+z^2)^{-3/2}\left[-x\bar{i} - y\bar{j} - z\bar{k}\right]$$

$$= -(x^2+y^2+z^2)^{-3/2}(\bar{r})$$

$$= -\bar{r}\cdot(x^2+y^2+z^2)^{-3/2} = -\bar{r}\cdot(r^2)^{-3/2}$$

$$= -\bar{r}\cdot r^{-3}$$

$$= \frac{\bar{r}}{r^3} = \cancel{\frac{1}{r^2}} \Rightarrow -\bar{r}\cdot r^{-3}$$

$2x^{-\frac{3}{2}}$

Directional derivative of $1/r$ in direction $\left(\nabla\left(\frac{1}{r}\right)\cdot\hat{e}\right)$

$$\hat{e} = \frac{\bar{r}}{|\bar{r}|}$$

$$\Rightarrow \frac{-\bar{r}}{r^3}\cdot\frac{\bar{r}}{|\bar{r}|} = \frac{-r^2}{|\bar{r}|\cdot r^3} = \frac{-1}{r\cdot|\bar{r}|} = \frac{-1}{r^2}$$

$|\bar{r}| = r$

$$\Rightarrow \frac{-1}{(x^2+y^2+z^2)} = \frac{-1}{(1)^2+(1)^2+(1)^2} = \frac{-1}{4+1+1} = \frac{-1}{6}$$

Divergence of Vector point Function:-

If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ is a continous differentiable by a vector point function, then

$$\nabla\bar{f} = \left(\bar{i}\cdot\frac{\partial}{\partial x} + \bar{j}\cdot\frac{\partial}{\partial y} + \bar{k}\cdot\frac{\partial}{\partial z}\right)\cdot\left(f_1\bar{i} + f_2\bar{j} + f_3\bar{k}\right)$$

$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$ is called divergence of $\bar{f}$ and is denoted by $\text{div}\bar{f}$

$$\boxed{\text{div}\bar{f} = \nabla\bar{f}}$$

physical interpretation of divergence:-

The divergence of a vector point function f gives at each point the rate per unit volume at which the physical Quantity is issuing from the point

Solenoidal vector:- A vector function is said to be solenoidal if $\text{div}\bar{f} = 0$.

* Evaluate divergence of $\bar{F} = 2x^2z\bar{i} + 3yz^2\bar{k} - xy^2z\bar{j}$ at point $(1,1,1)$

$$\bar{F} = (2x^2z)\bar{i} + (3yz^2)\bar{k} + (-xy^2z)\bar{j}$$

$$\bar{F} = F_1\bar{i} + F_3\bar{k} + F_2\bar{j}$$

$$\nabla\bar{F} = \frac{\partial}{\partial x}(2x^2z) + \frac{\partial}{\partial y}(-xy^2z) + \frac{\partial}{\partial z}(3yz^2)$$

$$= 2z(2x) + (-xz)(2y) + 3y(2z)$$

$$\text{div}\bar{F} = 4xz - 2xyz + 6zy.$$

$$(\nabla\bar{F})_{1,1,1} = 4(1)(1) - 2(1)(1) + 6(1)(1) = 8$$

* Find $\text{div}\bar{F}$ where $\bar{F} = \text{grad}[x^3+y^3+z^3-3xyz]$

$$\text{div}\bar{F} = \nabla\bar{F} = \text{grad}(x^3+y^3+z^3-3xyz)$$

$$\text{let } \phi = x^3+y^3+z^3-3xyz$$

$$\bar{F} = \text{grad}\phi = \bar{i}\cdot\frac{\partial\phi}{\partial x} + \bar{j}\cdot\frac{\partial\phi}{\partial y} + \bar{k}\cdot\frac{\partial\phi}{\partial z}$$

$$= \bar{i}\cdot\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz) + \bar{j}\cdot\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz) + \bar{k}\cdot\frac{\partial}{\partial z}(x^2+y^3+z^3-3xyz)$$

$$\bar{F} = \bar{i}(3x^2-3yz) + \bar{j}(3y^2-3xz) + \bar{k}(3z^2-3xy)$$

$$\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$$

$$\text{div}\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$$

$$\text{div}\bar{F} = 3(2x) + 6y + 6z = 6(x+y+z)$$

* If $\bar{F} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is solenoidal find p?

$$\bar{F} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$$

$$\text{div}\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$0 = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+pz)$$

$$0 = 1 + 3y\cdot\frac{\partial}{\partial x}(1) + 1 + p$$

$$0 = 1+1+p \Rightarrow \boxed{p=-2}$$

* Find $\text{div}\bar{F}$ where $\bar{F} = r^n\bar{r}$; find if it is solenoidal.
(or) prove that $r^n\bar{r}$ is solenoidal in $n=-3$ / P.T divergence

~~Given:~~ $\bar{F} =$ -of $r^n\bar{r} \Rightarrow \text{div}(r^n\bar{r}) = (n+3)r^n$ show that $\frac{\bar{r}}{r^3}$ is Solenoidal.

Given $\bar{F} = r^n\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$r = |\bar{r}| = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2$$

differentiate b.s w.r.t x.

$$\cancel{2}r\cdot\frac{\partial r}{\partial x} = \cancel{2}x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} ; \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{div}\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\bar{F} = r^n\cdot\bar{r} = r^n(x\bar{i} + y\bar{j} + z\bar{k})$$

$$\bar{F} = r^nx\bar{i} + r^ny\bar{j} + r^nz\bar{k}$$

$$\bar{F}_0 = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$$

$$F_1 = r^nx ; F_2 = r^ny ; F_3 = r^nz$$

$$\operatorname{div}\bar{f} = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= r^n(1) + \frac{\partial (r^n)}{\partial x}x + r^n(1) + \frac{\partial r^n}{\partial y}(y) + r^n(1) + \frac{\partial r^n}{\partial z}(z)$$

$$= r^n + n.r^{n-1}\frac{\partial r}{\partial x}.x + r^n + n.r^{n-1}\frac{\partial r}{\partial y}.y + r^n + n.r^{n-1}\frac{\partial r}{\partial z}.z$$

$$= 3r^n + n.r^{n-1}\left[\frac{\partial r}{\partial x}.x + \frac{\partial r}{\partial y}.y + \frac{\partial r}{\partial z}.z\right]$$

$$= 3r^n + nr^{n-1}\left[\frac{x}{r}.x + \frac{y}{r}.y + \frac{z}{r}.z\right]$$

$$= 3r^n + nr^{n-1}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right]$$

$$= 3r^n + nr^{n-1}\frac{1}{r}(x^2+y^2+z^2)$$

$$= 3r^n + nr^{n-1}\frac{1}{r}(r^2) = 3r^n + nr^{n-1}.r^{+1}$$

$$= 3r^n + n.r^n$$

$$= 3r^n + nr^n \Rightarrow r^n(3+n)$$

$$\operatorname{div}\bar{f} = 0 \Rightarrow (n+3)r^n = 0.$$

$$\boxed{n = -3}$$

$$\operatorname{div}(r^n.\bar{r}) = (n+3)r^n.$$

Hence proved.

Let $\bar{f} = r^n.\bar{r}$ be solenoidal then $\operatorname{div}\bar{f} = 0 \Rightarrow (n+3)r^n = 0$

$r^n = 0 \mid n+3=0$

$n = -3$

$\therefore \bar{f} = \bar{r}^{-3}\bar{r} = \frac{\bar{r}}{r^3}$ solenoidal

If $\phi = 2x^3y^2z^4$ show that $\operatorname{div}(\operatorname{grad}\phi) = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$.

Given $\phi = 2x^3y^2z^4$

$$\operatorname{grad}\phi = \bar{i}.\frac{\partial\phi}{\partial x} + \bar{j}.\frac{\partial\phi}{\partial y} + \bar{k}.\frac{\partial\phi}{\partial z}$$

$$= \bar{i}.\frac{\partial}{\partial x}(2x^3y^2z^4) + \bar{j}\frac{\partial}{\partial y}(2x^3y^2z^4) + \bar{k}.\frac{\partial}{\partial z}(2x^3y^2z^4)$$

$$= \bar{i}(2.3x^2y^2z^4) + \bar{j}(2x^3.2yz^4) + \bar{k}(2x^3y^2.4z^3)$$

$$\operatorname{grad}\phi = \bar{i}(6x^2y^2z^4) + \bar{j}(4x^3yz^4) + \bar{k}(8x^3y^2z^3)$$

$$= f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$$

$$\operatorname{div}(\operatorname{grad}\phi) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$= 6y^2z^4(2x) + 4x^3z^4(1) + 8x^3y^2(3z^2)$$

$$\operatorname{div}(\operatorname{grad}\phi) = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

* prove that $\operatorname{div}(\operatorname{grad} r^m) = m(m+1)r^{m-2}$

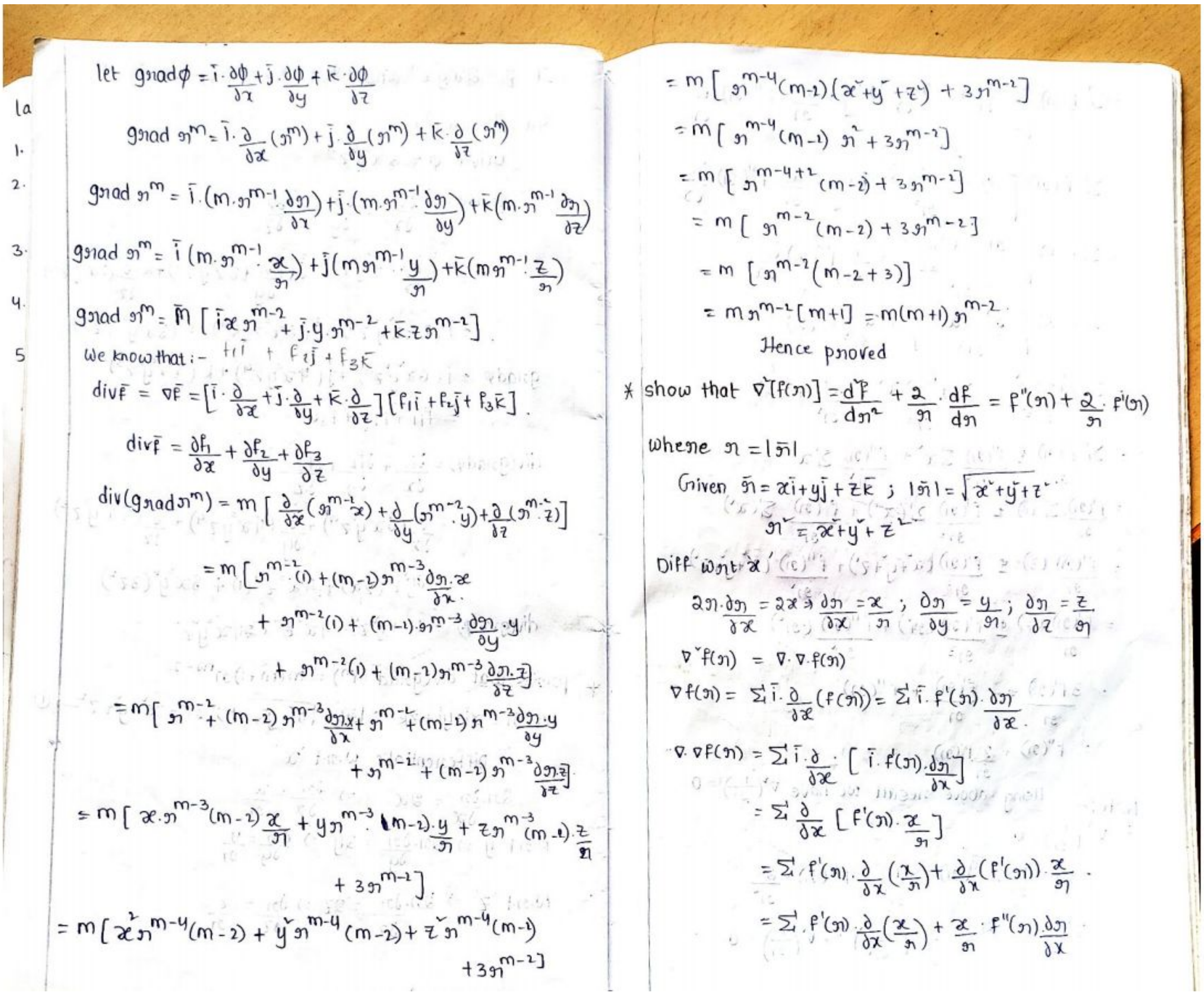
let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$; $|\bar{r}| = \sqrt{x^2+y^2+z^2}$; $r^2 = x^2+y^2+z^2 \longrightarrow ①$

Differentiate w.r.t 'x'

$$2r.\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

w.r.t 'y' $\Rightarrow 2r.\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$

w.r.t 'z' $\Rightarrow 2r.\frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$



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$$\text{let } \operatorname{grad}\phi = \bar{i}\cdot\frac{\partial\phi}{\partial x} + \bar{j}\cdot\frac{\partial\phi}{\partial y} + \bar{k}\cdot\frac{\partial\phi}{\partial z}$$

1. $$\operatorname{grad} r^m = \bar{i}\cdot\frac{\partial}{\partial x}(r^m) + \bar{j}\cdot\frac{\partial}{\partial y}(r^m) + \bar{k}\cdot\frac{\partial}{\partial z}(r^m)$$

2. $$\operatorname{grad} r^m = \bar{i}\cdot\left(m\cdot r^{m-1}\cdot\frac{\partial r}{\partial x}\right) + \bar{j}\cdot\left(m\cdot r^{m-1}\cdot\frac{\partial r}{\partial y}\right) + \bar{k}\left(m\cdot r^{m-1}\,\frac{\partial r}{\partial z}\right)$$

3. $$\operatorname{grad} r^m = \bar{i}\left(m\cdot r^{m-1}\cdot\frac{x}{r}\right) + \bar{j}\left(m r^{m-1}\cdot\frac{y}{r}\right) + \bar{k}\left(m r^{m-1}\cdot\frac{z}{r}\right)$$

4. $$\operatorname{grad} r^m = \bar{m}\left[\bar{i} x r^{m-2} + \bar{j}\cdot y\cdot r^{m-2} + \bar{k}\cdot z r^{m-2}\right]$$

5 We know that :- $f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\operatorname{div}\bar{F} = \nabla\bar{F} = \left[\bar{i}\cdot\frac{\partial}{\partial x} + \bar{j}\cdot\frac{\partial}{\partial y} + \bar{k}\cdot\frac{\partial}{\partial z}\right]\left[f_1\bar{i} + f_2\bar{j} + f_3\bar{k}\right].$$

$$\operatorname{div}\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\operatorname{div}(\operatorname{grad} r^m) = m\left[\frac{\partial}{\partial x}(r^{m-2}x) + \frac{\partial}{\partial y}(r^{m-2}\cdot y) + \frac{\partial}{\partial z}(r^{m-2}\cdot z)\right]$$

$$= m\Big[r^{m-2}(1) + (m-2)r^{m-3}\cdot\frac{\partial r}{\partial x}\cdot x$$

$$+ r^{m-2}(1) + (m-2)\cdot r^{m-3}\,\frac{\partial r}{\partial y}\cdot y$$

$$+ r^{m-2}(1) + (m-2)r^{m-3}\,\frac{\partial r}{\partial z}\cdot z\Big]$$

$$= m\Big[r^{m-2} + (m-2)r^{m-3}\frac{\partial r}{\partial x}x + r^{m-2} + (m-2)r^{m-3}\frac{\partial r}{\partial y}\cdot y$$

$$+ r^{m-2} + (m-2)r^{m-3}\frac{\partial r}{\partial z}z\Big]$$

$$= m\Big[x\cdot r^{m-3}(m-2)\frac{x}{r} + y r^{m-3}\cdot(m-2)\cdot\frac{y}{r} + z r^{m-3}(m-2)\cdot\frac{z}{r}$$

$$+ 3r^{m-2}\Big].$$

$$= m\Big[x^2 r^{m-4}(m-2) + y^2 r^{m-4}(m-2) + z^2 r^{m-4}(m-2)$$

$$+ 3r^{m-2}\Big]$$

$$= m\left[r^{m-4}(m-2)(x^2+y^2+z^2) + 3r^{m-2}\right]$$

$$= m\left[r^{m-4}(m-2)\, r^2 + 3r^{m-2}\right]$$

$$= m\left[r^{m-4+2}(m-2) + 3r^{m-2}\right]$$

$$= m\left[r^{m-2}(m-2) + 3r^{m-2}\right]$$

$$= m\left[r^{m-2}(m-2+3)\right]$$

$$= m r^{m-2}[m+1] = m(m+1)r^{m-2}.$$

Hence proved

* show that $\nabla^2[f(r)] = \frac{d^2f}{dr^2} + \frac{2}{r}\cdot\frac{df}{dr} = f''(r) + \frac{2}{r}\cdot f'(r)$

where $r = |\bar{r}|$

Given $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$; $|\bar{r}| = \sqrt{x^2+y^2+z^2}$

$$r^2 = x^2+y^2+z^2$$

Diff w.r.t. x

$$2r\cdot\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}\ ;\ \frac{\partial r}{\partial y} = \frac{y}{r}\ ;\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla^2 f(r) = \nabla\cdot\nabla\cdot f(r)$$

$$\nabla f(r) = \sum \bar{i}\cdot\frac{\partial}{\partial x}(f(r)) = \sum \bar{i}\cdot f'(r)\cdot\frac{\partial r}{\partial x}.$$

$$\nabla\cdot\nabla f(r) = \sum \bar{i}\cdot\frac{\partial}{\partial x}\left[\bar{i}\cdot f'(r)\cdot\frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r)\cdot\frac{x}{r}\right]$$

$$= \sum f'(r)\cdot\frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial x}(f'(r))\cdot\frac{x}{r}.$$

$$= \sum f'(r)\cdot\frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{x}{r}\cdot f''(r)\frac{\partial r}{\partial x}$$

$$= \sum f'(r)\left[\frac{r\cdot\frac{\partial}{\partial x}(x) - x\cdot\frac{\partial}{\partial x}(r)}{r^2}\right] + \frac{x}{r}\cdot f''(r)\cdot\frac{\partial r}{\partial x}.$$

$$= \sum f'(r)\left[\frac{r\cdot(1) - x\cdot \partial r/\partial x}{r^2}\right] + \frac{x}{r}\cdot f''(r)\cdot\frac{\partial r}{\partial x}.$$

$$= \sum f'(r)\left[\frac{\frac{r}{1} - x\cdot\frac{x}{r}}{r^2}\right] + \frac{x}{r} f''(r)\frac{\partial r}{\partial x}$$

$$= \sum f'(r)\left[\frac{r^2 - x^2}{r^3}\right] + \frac{x^2}{r^2} f''(r)$$

$$= \sum \frac{f'(r) r^2}{r^3} - \frac{\sum f'(r) x^2}{r^3} + \frac{x^2}{r^2} f''(r)$$

$$= \sum \frac{f'(r)}{r} \pm \frac{f'(r)}{r^3}\sum x^2 + \frac{f''(r)}{r^2}\sum x^2$$

$$= \frac{f'(r)}{r}\cdot\sum(1) \pm \frac{f'(r)}{r^3}\sum(x^2) + \frac{f''(r)}{r^2}\sum(x^2)$$

$$= \frac{f'(r)}{r}(3) \pm \frac{f'(r)}{r^3}(x^2+y^2+z^2) + \frac{f''(r)}{r^2}(x^2+y^2+z^2).$$

$$= \frac{f'(r)}{r}(3) \pm \frac{f'(r)}{r^3}(r^2) + \frac{f''(r)}{r^2}(r^2)$$

$$= \frac{3f'(r)}{r} \cancel{+} - \frac{f'(r)}{r} + f''(r)$$

$$= f''(r) + \frac{2}{r} f'(r)$$

Note:— Using above result we have $\nabla^2\left(\frac{1}{r}\right) = 0$.

(i) $\nabla^2\left(\frac{1}{r}\right) = 0$

$$f(r) = \frac{1}{r};\quad f'(r) = \frac{-1}{r^2};\quad f''(r) = \frac{2}{r^3}.$$

$$\Rightarrow \nabla^2\left(\frac{1}{r}\right) = \frac{2}{r^3} + \frac{2}{r}\cdot\left(\frac{-1}{r^2}\right) \Rightarrow \nabla^2\left(\frac{1}{r}\right) = 0$$

(ii) $\nabla^2(\log r) = \frac{1}{r^2}$

$$f(r) = \log r;\quad f'(r) = \frac{1}{r};\quad f''(r) = \frac{-1}{r^2}.$$

$$\nabla^2 \log r = \frac{-1}{r^2} + \frac{2}{r}\left(\frac{1}{r}\right) = \frac{-1}{r^2} + \frac{2}{r^2} = \frac{1}{r^2}.$$

* Using the line Integral calculate the work done by force $\bar{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$ along the line from (0,0,0) to (1,0,0) to (1,1,0) to (1,1,1)

$$W = \int_C \bar{F}\cdot d\bar{r}$$

$$W = ? \Rightarrow \bar{F} = (3x^2+6y)\bar{i} - (14z)\bar{j} + (20xz^2)\bar{k}$$

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$$

$$\bar{F}\cdot d\bar{r} = \left[(3x^2+6y)\bar{i} - (14z)\bar{j} + (20xz^2)\bar{k}\right]\cdot\left[dx\bar{i} + dy\bar{j} + dz\bar{k}\right]$$

$$\bar{F}\cdot d\bar{r} = (3x^2+6y)dx - (14z)(dy) + (20xz^2)dz$$

Along the line (0,0,0) to (1,0,0)

$y = 0;\ dy = 0 \mid z = 0;\ dz = 0$.

x changes from 0 to 1.

$$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_0^1 (3x^2+0)\,dx = \left[\frac{3x^3}{3}\right]_0^1 = \frac{3}{3} = 1$$

Along the line (1,0,0) to (1,1,0).

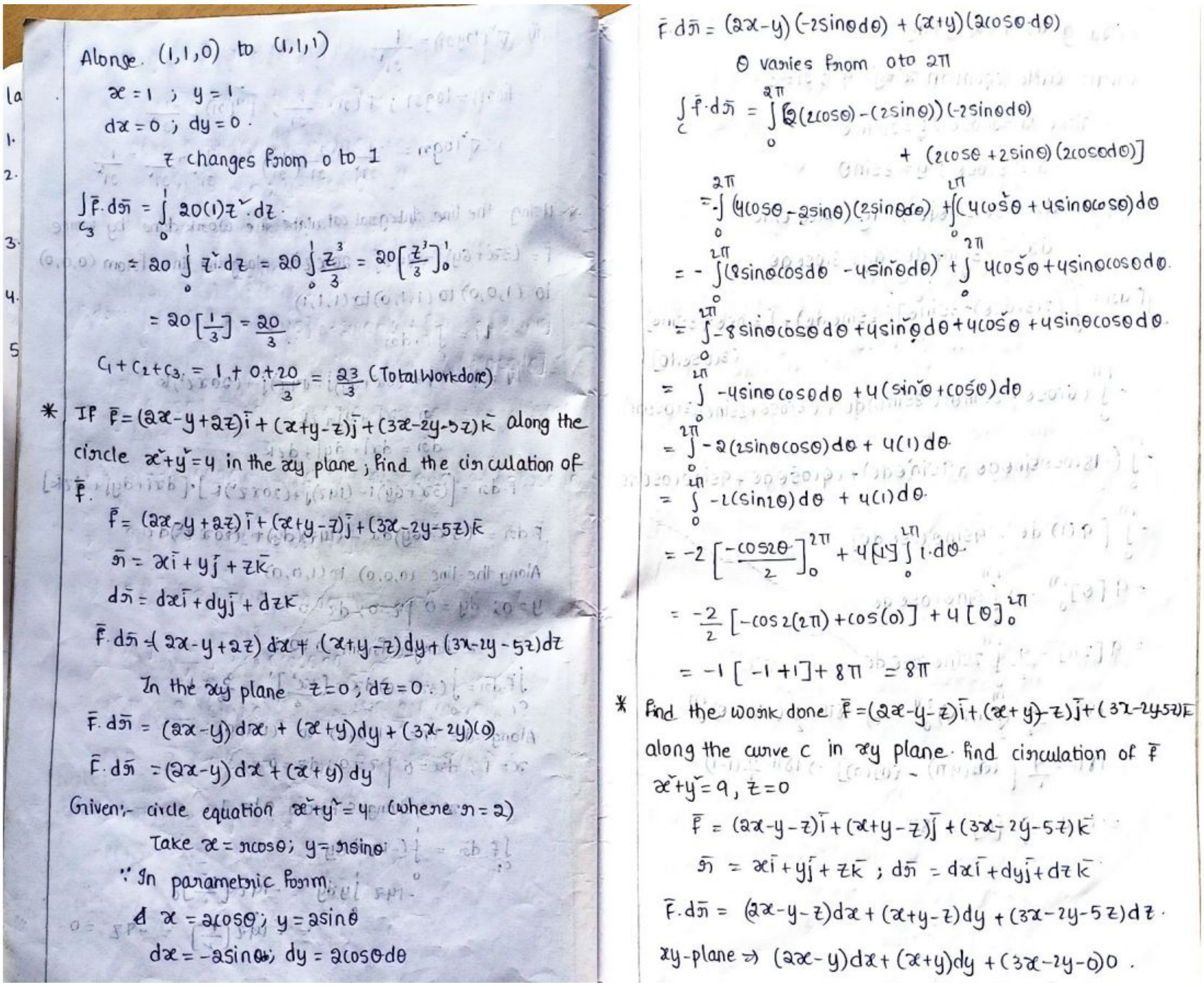
$x = 1;\ dx = 0 \mid z = 0;\ dz = 0$.

y changes from 0 to 1.

$$\int_{C_2} \bar{F}\cdot d\bar{r} = \int_0^1 (-14z)\,dy.$$

$$= -14z\int_0^1 y\,dy = -14z\left[\frac{y^2}{2}\right]_0^1$$

$$= -\overset{7}{14}z\left[\frac{1}{2}\right] = -7z = 0.$$



Alonge. (1,1,0) to (1,1,1)

$x = 1 \; ; \; y = 1$

$dx = 0 \; ; \; dy = 0$.

z changes from 0 to 1

$$\int_{C_3} \bar{F}\cdot d\bar{r} = \int_0^1 20(1)z^2 . dz$$

$$= 20\int_0^1 z^2 . dz = 20\int_0^1 \frac{z^3}{3} = 20\left[\frac{z^3}{3}\right]_0^1$$

$$= 20\left[\frac{1}{3}\right] = \frac{20}{3}$$

$C_1 + C_2 + C_3 = 1 + 0 + \frac{20}{3} = \frac{23}{3}$ (Total Workdone)

* If $\bar{F} = (2x - y + 2z)\bar{i} + (x + y - z)\bar{j} + (3x - 2y - 5z)\bar{k}$ along the circle $x^2 + y^2 = 4$ in the xy plane ; find the circulation of $\bar{F}$.

$\bar{F} = (2x - y + 2z)\bar{i} + (x + y - z)\bar{j} + (3x - 2y - 5z)\bar{k}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\bar{F}\cdot d\bar{r} = (2x - y + 2z)\,dx + (x + y - z)\,dy + (3x - 2y - 5z)\,dz$

In the xy plane $z = 0 \; ; \; dz = 0$.

$\bar{F}\cdot d\bar{r} = (2x - y)\,dx + (x + y)\,dy + (3x - 2y)(0)$

$\bar{F}\cdot d\bar{r} = (2x - y)\,dx + (x + y)\,dy$

Given:- circle equation $x^2 + y^2 = 4$ (where $r = 2$)

Take $x = r\cos\theta \; ; \; y = r\sin\theta$

∵ In parametric form

$x = 2\cos\theta \; ; \; y = 2\sin\theta$

$dx = -2\sin\theta \; ; \; dy = 2\cos\theta\, d\theta$

$\bar{F}\cdot d\bar{r} = (2x - y)(-2\sin\theta\, d\theta) + (x + y)(2\cos\theta . d\theta)$

θ varies from 0 to 2π

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi} [(2(2\cos\theta) - (2\sin\theta))(-2\sin\theta\, d\theta)$$
$$+ (2\cos\theta + 2\sin\theta)(2\cos\theta\, d\theta)]$$

$$= -\int_0^{2\pi} (4\cos\theta - 2\sin\theta)(2\sin\theta\, d\theta) + \int_0^{2\pi} (4\cos^2\theta + 4\sin\theta\cos\theta)\, d\theta$$

$$= -\int_0^{2\pi} (8\sin\theta\cos\theta\, d\theta - 4\sin^2\theta\, d\theta) + \int_0^{2\pi} 4\cos^2\theta + 4\sin\theta\cos\theta\, d\theta.$$

$$= \int_0^{2\pi} -8\sin\theta\cos\theta\, d\theta + 4\sin^2\theta\, d\theta + 4\cos^2\theta + 4\sin\theta\cos\theta\, d\theta.$$

$$= \int_0^{2\pi} -4\sin\theta\cos\theta\, d\theta + 4(\sin^2\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} -2(2\sin\theta\cos\theta)\, d\theta + 4(1)\, d\theta$$

$$= \int_0^{2\pi} -2(\sin 2\theta)\, d\theta + 4(1)\, d\theta$$

$$= -2\left[\frac{-\cos 2\theta}{2}\right]_0^{2\pi} + 4[1]\int_0^{2\pi} 1 . d\theta$$

$$= \frac{-2}{2}\left[-\cos 2(2\pi) + \cos(0)\right] + 4\,[\theta]_0^{2\pi}$$

$$= -1\,[-1 + 1] + 8\pi = 8\pi$$

* Find the work done $\bar{F} = (2x - y - z)\bar{i} + (x + y - z)\bar{j} + (3x - 2y - 5z)\bar{k}$ along the curve c in xy plane. find circulation of $\bar{F}$ $x^2 + y^2 = 9,\; z = 0$

$\bar{F} = (2x - y - z)\bar{i} + (x + y - z)\bar{j} + (3x - 2y - 5z)\bar{k}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \; ; \; d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\bar{F}\cdot d\bar{r} = (2x - y - z)\,dx + (x + y - z)\,dy + (3x - 2y - 5z)\,dz$.

xy-plane ⇒ $(2x - y)\,dx + (x + y)\,dy + (3x - 2y - 0)0$.

$(2x-y)dx + (x+y)dy$

la Given:- circle equation $x^2+y^2=9$; $r=3$.

1. Take $x = r\cos\theta$; $y = r\sin\theta$

2. $x = 3\cos\theta$; $y = 3\sin\theta$

$dx = 3(-\sin\theta)d\theta$; $dy = 3(\cos\theta)d\theta$

3. $dx = -3\sin\theta\, d\theta$; $dy = 3\cos\theta\, d\theta$

4. $$\int_c \bar{F}\cdot d\bar{r} = \int_0^{2\pi} [2(3\cos\theta) - 3\sin\theta](-3\sin\theta\, d\theta) + [3\cos\theta + 3\sin\theta](3\cos\theta d\theta]$$

5

$$= \int_0^{2\pi} (6\cos\theta - 3\sin\theta)(-3\sin\theta)d\theta + (3\cos\theta + 3\sin\theta)(3\cos\theta d\theta)$$

$$= \int_0^{2\pi} (-18\cos\theta\sin\theta d\theta + 9\sin^2\theta d\theta) + (9\cos^2\theta d\theta + 9\sin\theta\cos\theta d\theta)$$

$$= \int_0^{2\pi} [9(1)\, d\theta - 9\sin\theta\cos\theta d\theta].$$

$$= 9[\theta]_0^{2\pi} - 9\int_0^{2\pi} \sin\theta\cos\theta\, d\theta$$

$$= 9[2\pi] - \frac{9}{2}\int_0^{2\pi} \underline{2\sin\theta\cos\theta\, d\theta}$$

$$= 18\pi - \frac{9}{2}\cdot(\int_0^{2\pi}\sin 2\theta)_0^{2\pi} \Rightarrow 18\pi - \frac{9}{2}(\cos 2\theta)_0^{2\pi}$$

$$= 18\pi - \frac{9}{2}[\underset{1}{\cos(4\pi)} - \underset{1}{\cos(0)}] \Rightarrow 18\pi - \frac{9}{2}(1-1)$$

$$= 18\pi - \frac{9}{2}[0] = 18\pi$$

If $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ evaluate $\int_c \bar{F}\cdot d\bar{r}$ where curve c is rectangle in xy plane. bounded by $y=0, y=b, x=0, x=a$

Given:- $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$; $d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\bar{F}\cdot d\bar{r} = ((x^2+y^2)\bar{i} - 2xy\bar{j})\cdot(dx\bar{i} + dy\bar{j} + dz\bar{k})$

$\bar{F}\cdot d\bar{r} = (x^2+y^2)dx - (2xy)dy$

Here rectangle is in xy plane

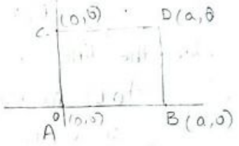
$x=0, x=a$; $y=0, y=b$

$(x,y) = (0,0)$

$(x,y) = (0,b)$

$(x,y) = (a,0)$

$(x,y) = (a,b)$



Along the line AB

$A(0,0)$ $B(a,0)$

$y=0$; $dy=0$

x changes from 0 to a.

$$\int_{C_1} \bar{F}\cdot d\bar{r} \Rightarrow \int_0^a ((x^2+y^2)dx - (2xy)dy)d$$

$\bar{F}\cdot d\bar{r} = (x^2)dx$

$$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_0^a (x^2)dx = \int_0^a \frac{x^3}{3} = \frac{a^3}{3}$$

Along the line BD.

$B(a,0)$ $D(a,b)$

$x=a$; $dx=0$.

y changes from 0 to b.

$$\int_{C_2} \bar{F}_2\cdot d\bar{r} = \int_0^b (-2ay)dy = -2a\left[\frac{y^2}{2}\right]_0^b = -\frac{2a}{2}(b^2] = -ab^2$$

Along the line DC

$D(a,b)$ $C(0,b)$

$y = b ; dy = 0$

x changes from a to 0

$$\int_{C_3} \bar{F}\cdot d\bar{r} = \int_a^0 (x^2+b^2)\,dx$$

$$= \int_a^0 x^2\,dx + \int_a^0 b^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^a + b^2[x]_0^a \quad = \left[\frac{x^3}{3}\right]_a^0 + b^2[x]_a^0$$

$$= \frac{a^3}{3} + b^2[a] \quad = \frac{-a^3}{3} - ab^2$$

Along the line CA

$C(0,b)$ $A(0,0)$.

$x = 0 ; dx = 0$.

y changes from b to 0.

$$\int_{C_4} \bar{F}\cdot d\bar{r} = \int_b^0 [0] = 0.$$

(-1,-1)
(1,-1)
(-1,1)
(1,1)

$$\int_C \bar{F}\cdot d\bar{r} = \int_{C_1} \bar{F}\cdot d\bar{r} + \int_{C_2} \bar{F}\cdot d\bar{r} + \int_{C_3} \bar{F}\cdot d\bar{r} + \int_{C_4} \bar{F}\cdot d\bar{r}$$

$$= \frac{\cancel{a^3}}{3} - ab^2 - \frac{\cancel{a^3}}{3} - ab^2 + 0$$

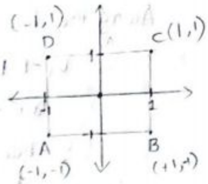
$$= -2ab^2$$

* Evaluate the line integral $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$ where C is a square formed by lines $x = \pm 1 ; y = \pm 1$

$\bar{F}\cdot d\bar{r} = (x^2+xy)\,dx + (x^2+y^2)\,dy$ $\quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$x = -1, 1 ; y = -1, 1$ $\quad (b<a)$

Along the line AB



$A(-1,-1)$ $B(1,-1)$

$y = -1 ; dy = 0$

x changes from -1 to 1

$$\int_{C_1} F\cdot d\bar{r} = \int_{-1}^{1} (x^2 + x(-1))\,dx$$

$$= \int_{-1}^{1} (x^2 - x)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1}$$

$$= \left[\frac{1}{3} - \frac{1}{2}\right] - \left[\frac{-1}{3} - \frac{1}{2}\right] = \frac{1}{3} - \frac{1}{2} + \frac{1}{3} + \frac{1}{2} = \frac{2}{3}$$

Along the line BC ; $B(1,-1)$ $C(1,1)$

$x = 1 ; dx = 0$

y changes from -1 to 1

$$\int_{C_2} \bar{F}\cdot d\bar{r} = \int_{-1}^{1} (1+y^2)\,dy = \int_{-1}^{1} (1+y^2)\,dy = \left[y + \frac{y^3}{3}\right]_{-1}^{1}$$

$$= \left[1 + \frac{1}{3}\right] - \left[(-1) + \frac{(-1)}{3}\right] = 1 + \frac{1}{3} + 1 + \frac{1}{3} = \frac{8}{3}$$

Along the line CD ; $C(1,1)$ $D(-1,1)$

$y = 1 ; dy = 0$

x changes from 1 to -1

$$\int_{C_3} \bar{F}\cdot d\bar{r} = \int_{1}^{-1} (x^2+x)\,dx = -\int_{-1}^{1} (x^2+x)\,dx$$

$$= -\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1}$$

$$= -\left[\left(\frac{1}{3} + \frac{1}{2}\right) - \left(\frac{-1}{3} + \frac{1}{2}\right)\right] = -\left[\frac{1}{3} + \cancel{\frac{1}{2}} + \frac{1}{3} - \cancel{\frac{1}{2}}\right]$$

$$= \frac{-2}{3}$$

Along the line DA

$D(-1,1) \quad A(-1,-1)$

$x = -1 \; ; \; dx = 0$

y changes from 1 to -1

$$\int_{C_4} \bar{F}\cdot d\bar{r} = \int_{1}^{-1} (1+y^2)\,dy = -\int_{-1}^{1} (1+y^2)\,dy$$

$$= -\left[y + \frac{y^3}{3}\right]_{-1}^{1}$$

$$= -\left[\left[1+\frac{1}{3}\right] - \left[-1 + \frac{(-1)}{3}\right]\right]$$

$$= -\left[1+\frac{1}{3} + 1 + \frac{1}{3}\right] = -\left[2+\frac{2}{3}\right] = -\frac{8}{3}$$

$$C = \int_{C_1} \bar{f}\,d\bar{r} + \int_{C_2} \bar{f}\,d\bar{r} + \int_{C_3} \bar{f}\,d\bar{r} + \int_{C_4} \bar{f}\,d\bar{r}$$

$$= \frac{2}{3} + \frac{8}{3} - \frac{2}{3} - \frac{8}{3}$$

$$= 0.$$

Vector identities:- If $\bar{a}$ & ϕ are differentiable vector and scalarpoint function then

prove that $\operatorname{div}(\phi\bar{a}) = (\operatorname{grad}\phi)\,\bar{a} + \phi \operatorname{div}\bar{a}$

$$\operatorname{div}(\phi\bar{a}) = \nabla(\phi\bar{a})$$

$$= \left(\bar{i}\cdot\frac{\partial}{\partial x} + \bar{j}\cdot\frac{\partial}{\partial y} + \bar{k}\cdot\frac{\partial}{\partial z}\right)(\phi\bar{a})$$

$$= \bar{i}\frac{\partial}{\partial x}(\phi\bar{a}) + \bar{j}\cdot\frac{\partial}{\partial y}(\phi\bar{a}) + \bar{k}\cdot\frac{\partial}{\partial z}(\phi\bar{a})$$

$$= \bar{i}\left[\frac{\partial\phi}{\partial x}\bar{a} + \frac{\partial\bar{a}}{\partial x}\cdot\phi\right] + \bar{j}\left[\frac{\partial\phi}{\partial y}\bar{a} + \phi\cdot\frac{\partial\bar{a}}{\partial y}\right] + \bar{k}\left[\frac{\partial\phi}{\partial z}\bar{a} + \phi\cdot\frac{\partial\bar{a}}{\partial z}\right]$$

$$= \bar{i}\,\frac{\partial\phi}{\partial x}\bar{a} + \bar{j}\cdot\frac{\partial\phi}{\partial y}\bar{a} + \bar{k}\cdot\frac{\partial\phi}{\partial z}\bar{a} + \bar{i}\phi\frac{\partial\bar{a}}{\partial x} + \bar{j}\cdot\phi\frac{\partial\bar{a}}{\partial y} + \bar{k}\,\phi\cdot\frac{\partial\bar{a}}{\partial z}$$

$$= \left[\bar{i}\cdot\frac{\partial\phi}{\partial x} + \bar{j}\cdot\frac{\partial\phi}{\partial y} + \bar{k}\cdot\frac{\partial\phi}{\partial z}\right]\bar{a} + \phi\left[\bar{i}\cdot\frac{\partial\bar{a}}{\partial x} + \bar{j}\cdot\frac{\partial\bar{a}}{\partial y} + \bar{k}\,\frac{\partial\bar{a}}{\partial z}\right]$$

$$= (\operatorname{grad}\phi)\,\bar{a} + \left[\bar{i}\cdot\frac{\partial}{\partial x} + \bar{j}\cdot\frac{\partial}{\partial y} + \bar{k}\cdot\frac{\partial}{\partial z}\right]\bar{a}$$

$$= (\operatorname{grad}\phi)\,\bar{a} + \phi(\nabla\bar{a}) = (\operatorname{grad}\phi)\,\bar{a} + \phi \operatorname{div}\bar{a}$$

prove that $\bar{\nabla}\times(\bar{A}\times\bar{B}) = (\nabla\bar{B})\bar{A} - (\nabla\cdot\bar{A})\bar{B} + (\bar{B}\cdot\nabla)\bar{A} - (\bar{A}\cdot\nabla)\bar{B}$

$$\nabla\times(\bar{A}\times\bar{B}) = \Sigma\,\bar{i}\times\frac{\partial}{\partial x}(\bar{A}\times\bar{B})$$

$$= \Sigma\,\bar{i}\times\left[\frac{\partial\bar{A}}{\partial x}\times\bar{B} + \bar{A}\times\frac{\partial\bar{B}}{\partial x}\right]$$

$$= \Sigma\,\bar{i}\times\frac{\partial\bar{A}}{\partial x}\times\bar{B} + \Sigma\,\bar{i}\times\bar{A}\times\frac{\partial\bar{B}}{\partial x}$$

$$\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$$

$$\bar{i}\times\frac{\partial\bar{A}}{\partial x}\times\bar{B} = (\bar{i}\cdot\bar{B})\frac{\partial\bar{A}}{\partial x} - \left(\bar{i}\cdot\frac{\partial\bar{A}}{\partial x}\right)\bar{B}$$

$$\bar{i}\times\bar{A}\times\frac{\partial\bar{B}}{\partial x} = \left(\bar{i}\cdot\frac{\partial\bar{B}}{\partial x}\right)\cdot\bar{A} - (\bar{i}\cdot\bar{A})\cdot\frac{\partial\bar{B}}{\partial x}$$

$$\nabla\times(\bar{A}\times\bar{B}) = \Sigma\left(\bar{i}\cdot\frac{\partial\bar{B}}{\partial x}\right)\bar{A} -$$

$$= \Sigma\,(\bar{i}\cdot\bar{B})\frac{\partial\bar{A}}{\partial x} - \Sigma\left(\bar{i}\cdot\frac{\partial\bar{A}}{\partial x}\right)\bar{B} + \Sigma\left(\bar{i}\cdot\frac{\partial\bar{B}}{\partial x}\right)\bar{A} - \Sigma(\bar{i}\cdot\bar{A})\frac{\partial\bar{B}}{\partial x}$$

$$= \bar{B}\,\Sigma\left(\bar{i}\cdot\frac{\partial\bar{A}}{\partial x}\right)\bar{A} - \bar{B}\cdot\Sigma\left(\bar{i}\cdot\frac{\partial}{\partial x}\right)\bar{A} + \bar{A}\cdot\Sigma\left(\bar{i}\cdot\frac{\partial}{\partial x}\right)\bar{B} - \bar{A}\cdot\Sigma\left(\bar{i}\cdot\frac{\partial}{\partial x}\right)\bar{B}$$

$$= (\bar{B}\nabla)\bar{A} - \bar{B}(\nabla)\bar{A} + \bar{A}(\nabla)\bar{B} - \bar{A}(\nabla)\bar{B}$$

$$= (\nabla\bar{B})\bar{A} - (\nabla\bar{A})\bar{B} + (\nabla\bar{B})\bar{A} - (\bar{A}\cdot\nabla)\bar{B}$$

Unit-2

Laplace transforms

Definition

Let $f(t)$ be a given function and defined for all positive values of 't'

Then Laplace transforms of $f(t)$ is denoted by $L[f(t)]$ (or) $F(s)$ defined by

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

where 's' is a real (or) complex number.

Sufficient condition for existance of Laplace transform

① $f(t)$ must be piecewise continuous function

② The function $f(t)$ is of exponential order 'a'

ie $\underset{t\to\infty}{Lt}\, e^{-at} f(t) =$ finite quantity.

Ex The function $f(t) = t^2$ is of exponential order '3'

$$\underset{t\to\infty}{Lt}\, e^{-3t} f(t) = \underset{t\to\infty}{Lt}\, e^{-3t} t^2 = \underset{t\to\infty}{Lt}\, \frac{t^2}{e^{3t}} = \left(\frac{\infty}{\infty}\right)$$

$$= \underset{t\to\infty}{Lt}\, \frac{2t}{3e^{3t}} \quad \text{by L' Hospital Rule}$$

$$= \underset{t\to\infty}{Lt}\, \frac{2}{9e^{3t}} = \frac{2}{\infty} = 0 = \text{finite quantity.}$$

Hence $f(t) = t^2$ is of exponential order '3'.

Note $f(t) = t^2 \sin at$, e^{at} etc are all of exponential order and also continuous, But $f(t) = e^{t^3}, e^{t^2}$ is not exponential order and such that it's Laplace transforms does not exist.

Laplace transforms of some standard functions

① $L[c] = c/s$

② $L[1] = 1/s$

③ $L[e^{at}] = 1/s-a$

④ $L[e^{-at}] = \frac{1}{s+a}$

⑤ $L[\sin at] = a/s^2+a^2$

⑥ $L[\cos at] = s/s^2+a^2$

⑦ $L[\sinh at] = a/s^2-a^2$

⑧ $L[\cosh at] = \frac{s}{s^2-a^2}$

⑨ $L[t^n] = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}$

⑩ $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$

⑪ $\Gamma(1) = 1,\ \Gamma(1/2) = \sqrt{\pi},\ \Gamma(n+1) = n!$

$\Gamma(n+1) = n\,\Gamma(n)$

⑫ $L[c_1 f_1(t) + c_2 f_2(t)] = c_1 L[f_1(t)] + c_2 L[f_2(t)]$

① Find Laplace transforms of $e^{3t} - 2e^{-2t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9$

sol $L\left(e^{3t} - 2e^{-2t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9\right]$

$$= L[e^{3t}] - 2L[e^{-2t}] + L[\sin 2t] + L[\cos 3t]$$

$$+ L[\sinh 3t] - 2L[\cosh 4t] + L[9]$$

$$= \frac{1}{s-3} - 2\frac{1}{s+2} + \frac{2}{s^2+4} + \frac{s}{s^2+9}$$

$$+ \frac{3}{s^2-3^2} - 2\frac{s}{s^2-4^2} + \frac{9}{s} = F(s)$$

② $f(t) = \begin{cases} 1 ; & \text{If } 0<t<1 \\ 0 ; & \text{If } 1<t<2 \\ 5 & \text{If } 2<t<3 \\ 0 & \text{If } t>3 \end{cases}$ Find $L[f(t)]$

sol $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^1 e^{-st}\cdot 1\,dt + \int_1^2 e^{-st}\cdot 0\,dt + \int_2^3 e^{-st}\cdot 5\,dt$$

$$+ \int_3^\infty e^{-st}\cdot 0\,dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_0^1 + 0 + 5\left(\frac{e^{-st}}{-s}\right)_2^3 + 0$$

$$= \frac{e^{-s}}{-s} - \frac{1}{-s} + 5\left(\frac{e^{-3s}}{-s} - \frac{e^{-2s}}{-s}\right)$$

$$= \frac{1-e^{-s}}{s} - 5\frac{(e^{-3s} - e^{-2s})}{s}$$

$$L[f(t)] = \frac{1}{s}\left(1 - e^{-s} + 5e^{-2s} - 5e^{-3s}\right)$$

③ Find $L[\cos 3t \sin 5t] = L\left[\frac{1}{2} 2\sin 5t \cos 3t\right]$

$$= \frac{1}{2} L[2\sin 5t \cos 3t]$$

$$= \frac{1}{2} L[\sin(5t+3t) + \sin(5t-3t)]$$

$$= \frac{1}{2}\left[L[\sin 8t] + L[\sin 2t]\right]$$

$$L[\cos 3t \sin 5t] = \frac{1}{2}\left[\frac{8}{s^2+8^2} + \frac{2}{s^2+2^2}\right]$$

④ $L[(\sin t - \cos t)^3]$

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$= L[(\sin^3 t - \cos^3 t) - 3(1-\cos^2 t)\cos t + 3\sin t\,(1-\sin^2 t)$

$= L[3(\sin t - \cos t) - 2(\sin^3 t - \cos^3 t)]$

$= L[3(\sin t - \cos t) - 2[\frac{1}{4}(\sin 3t + 3\sin t) - \frac{1}{4}(\cos 3t + 3\cos t)]$

$= L[\frac{1}{2}(3\sin t - 3\cos t + \sin 3t + \cos 3t)]$

$= \frac{1}{2}\left[\frac{3}{s^2+1} - \frac{3s}{s^2+1} + \frac{3}{s^2+9} + \frac{s}{s^2+9}\right] = L[(\sin t - \cos t)^3]$

⑤ find $L[\cos t \cos 2t \cos 3t] = L[\frac{1}{2}\cos t\, 2\cos 2t \cos 3t)]$

$= \frac{1}{2}L[\cos t(\cos 5t + \cos t)]$

$= \frac{1}{2}L[\cos t \cos 5t + \cos^2 t)]$

$= \frac{1}{4}L[\cos 6t + \cos 4t + 1 + \cos 2t]$

$= \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2+4} + \frac{s}{s^2+16} + \frac{s}{s^2+36}\right)$

⑥ $L[\sin(\omega t+\alpha)] = L(\sin\omega t\cos\alpha + \cos\omega t\sin\alpha)$

$L[\sin(\omega t+\alpha)] = \cos\alpha \cdot \frac{\omega}{s^2+\omega^2} + \sin\alpha \cdot \frac{s}{s^2+\omega^2}$

⑦ $L[t^2+at+b] = L[t^2] + aL[t] + L[b]$

$= \frac{2!}{s^3} + a \cdot \frac{1!}{s^2} + b/s$

$L[t^2+at+b] = \frac{2}{s^3} + \frac{a}{s^2} + b/s$

First Translation theorem (or First shifting theorem)

Statement

If $L[f(t)] = F(s)$ then $L[e^{at} f(t)] = F(s-a)$

Ex find $L[e^{-t}(3\sin 2t - 5\cosh 2t]$

$= L[3e^{-t}\sin 2t] - L[5e^{-t}\cosh 2t]$

$= 3L[e^{-t}\sin 2t] - 5L[e^{-t}\cosh 2t]$

$= 3\,\frac{2}{(s+1)^2+2^2} - 5 \cdot \frac{s}{(s+1)^2-2^2}$

$= \frac{6}{(s^2+2s+5)} - \frac{5(s+1)}{s^2+2s-3}$

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Ex Find $L[e^{3t}\sin^2 t]$

Sol $L[\sin^2 t] = L\left[\frac{1-\cos 2t}{2}\right]$

$= \frac{1}{2} L[1-\cos 2t]$

$= \frac{1}{2}\left[L[1] - L[\cos 2t]\right]$

$L[\sin^2 t] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$L[e^{3t}\sin^2 t] = \frac{1}{2}\left[\frac{1}{s-3} - \frac{s-3}{(s-3)^2+4}\right]$

Ex $L[\cosh at \sin bt] = L\left[\frac{e^{at}+e^{-at}}{2}\sin bt\right]$

$= \frac{1}{2} L\left[e^{at}\sin bt + e^{-at}\sin bt\right]$

$L[\cosh at \sin bt] = \frac{1}{2}\left[\frac{b}{(s-a)^2+b^2} + \frac{b}{(s+a)^2+b^2}\right]$

Ex Find $L[e^{4t}\sin t\cos 2t] = L\left[\frac{1}{2}e^{4t}\, 2\cos 2t\sin t\right]$

$= \frac{1}{2} L[e^{4t}(\sin 3t - \sin t)]$

$= \frac{1}{2}\left[L[e^{4t}\sin 3t] - L[e^{4t}\sin t]\right]$

$L[e^{4t}\sin t\cos 2t] = \frac{1}{2}\left[\frac{3}{(s-4)^2+3^2} - \frac{1}{(s-4)^2+1^2}\right]$

Second translation theorem (or) second shifting theorem

Statement

If $L[f(t)] = F(s)$ and $g(t) = \begin{cases} f(t-a) ; & t > a \\ 0 ; & t < a \end{cases}$

Then $L[g(t)] = e^{-as}F(s)$

proof LHS $L[g(t)] = \int_0^\infty e^{-st} g(t)\, dt$

$= \int_0^a e^{-st}. 0\, dt + \int_a^\infty e^{-st} f(t-a)\, dt$

$= 0 + \int_a^\infty e^{-st} f(t-a)\, dt$

put $t-a = x$

$\Rightarrow dt = dx$

when $t = a \Rightarrow x = 0$

when $t = \infty \Rightarrow x = \infty$

$$L(g(t)) = \int_{x=0}^{\infty} e^{-s(x+a)} f(x)\,dx$$

$$= e^{-as} \int_{0}^{\infty} e^{-sx} f(x)\,dx$$

$$L[g(t)] = e^{-as} F(s)$$

Another form of Second Shifting theorem

statement

If $L[f(t)] = F(s)$ then

$L[f(t-a)H(t-a)] = e^{-as} F(s)$ where

$H(t) = \begin{cases} 1 & \text{If } t>0 \\ 0 & \text{If } t<0 \end{cases}$ and $H(t)$ is called Heaviside unit step function.

Ex Find Laplace transform of $g(t)$

where $g(t) = \begin{cases} \cos(t-2\pi/3), & \text{If } t>2\pi/3 \\ 0 ; & t<2\pi/3 \end{cases}$

Sol let $f(t) = \cos t$

$L[f(t)] = L[\cos t] = \frac{s}{s^2+1} = F(s)$

Now $g(t) = \begin{cases} f(t-2\pi/3) ; & t>2\pi/3 \\ 0 ; & t<2\pi/3 \end{cases}$

Applying second shifting theorem

$$L[g(t)] = e^{-as} F(s)$$

$$L[g(t)] = e^{-\frac{2\pi}{3}s} \frac{s}{s^2+1} = \frac{s e^{-2\pi/3 s}}{s^2+1}$$

Ex Find Laplace transform of $3\cos 4(t-2)u(t-2)$

Sol let $f(t) = 3\cos 4t$

Then $L[f(t)] = L[3\cos 4t] = 3L[\cos 4t] = \frac{3s}{s^2+4^2}$

By second shifting theorem $L[f(t-2)u(t-2)] = e^{-2s}F(s)$

$$L[3\cos 4(t-2)u(t-2)] = e^{-2s}\frac{3s}{s^2+16} = \frac{3se^{-2s}}{s^2+16}.$$

$L[3\cos 4(t-2)u(t-2)]$

Change of Scale property

If $L[f(t)] = F(s)$ then $L[f(at)] = \frac{1}{a} F(s/a)$

Ex. If $L[f(t)] = \frac{9s^2 - 12s + 15}{(s-1)^3}$ find $L[f(3t)]$ using change of scale property.

Sol Given $L[f(t)] = \frac{9s^2 - 12s + 15}{(s-1)^3} = F(s)$

By change of scale property

$$L[f(3t)] = \frac{1}{3} F(s/3) = 1/3 \, \frac{9(s/3)^2 - 12(s/3) + 15}{(s/3 - 1)^3}$$

$$\therefore L[f(3t)] = \frac{9(s^2 - 4s + 15)}{(s-3)^3}$$

Laplace Transform of derivatives :-

If $L[f(t)] = F(s)$ then

$L[f'(t)] = sL[f(t)] - f(0)$

$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$

$L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$

$L[f^{iv}(t)] = s^4 L[f(t)] - s^3 f(0) - s^2 f'(0) - s f''(0) - f'''(0)$

Laplace transform of integrals

Statement

If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

Ex If $L[f(t)] = \frac{9s^2-12s+15}{(s-1)^3}$ find $L[f(3t)]$ using change of scale property.

sol Given $L[f(t)] = \frac{9s^2-12s+15}{(s-1)^3} = F(s)$

By change of scale property

$$L[f(3t)] = \frac{1}{3}F(s/3) = 1/3\ \frac{9(s/3)^2-12(s/3)+15}{(s/3-1)^3}$$

$$\therefore L[f(3t)] = \frac{9(s^2-6s+15)}{(s-3)^3}$$

Laplace Transform of derivatives :-

If $L[f(t)] = F(s)$ then

$L[f'(t)] = sL[f(t)] - f(0)$

$L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0)$

$L[f'''(t)] = s^3L[f(t)] - s^2f(0) - sf'(0) - f''(0)$

$L[f^{IV}(t)] = s^4L[f(t)] - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)$

Laplace transform of integrals

statement

If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

Ex find $L\left[\int_0^t e^{-t}\cos t\,dt\right]$

sol let $f(t) = e^{-t}\cos t$

$L[f(t)] = L[e^{-t}\cos t]$

$= \frac{s+1}{(s+1)^2+1^2} = \frac{s+1}{s^2+2s+2}$ $\left(\because L[\cos t] = \frac{s}{s^2+1}\right)$

$$\therefore L\left[\int_0^t e^{-t}\cos t\,dt\right] = \frac{F(s)}{s} = \frac{1}{s}\,\frac{s+1}{s^2+2s+2} = \frac{s+1}{s(s^2+2s+2)}$$

multiplication by t.

If $L[f(t)] = F(s)$ then $L[t^n f(t)] = (-1)^n\frac{d^n}{ds^n}L[f(t)]$

Ex Find $L[t\,e^{2t}\cos 2t]$

sol $L[\cos 2t] = s/s^2+4$

$L[e^{2t}\cos 2t] = s-2/(s-2)^2+4 = s-2/s^2-4s+8$

$L[t\,e^{-2t}\cos 2t] =$

$$L[te^{2t}\cos 2t] = (-1)^1 \frac{d^1}{ds^1}\left(\frac{s-2}{s^2-4s+8}\right)$$

$$= -\left(\frac{1(s^2-4s+8)-(s-2)(2s-4)}{(s^2-4s+8)^2}\right)$$

$$= -\frac{\left(s^2-4s+8-(2s^2-4s-4s+8)\right)}{(s^2-4s+8)^2}$$

$$\therefore L[te^{2t}\cos 2t] = -\frac{(-s^2+4s)}{(s^2-4s+8)^2} = \frac{s(s-4)}{(s^2-4s+8)^2}$$

Ex find $L[t^3e^{2t}\sin t]$

Sol $L\{\sin t\} = \frac{1}{s^2+1}$

$$L[e^{2t}\sin t] = \frac{1}{(s-2)^2+1} = \frac{1}{s^2-4s+5}$$

$$L[t^3e^{2t}\sin t] = (-1)^3\frac{d^3}{ds^3}\frac{1}{s^2-4s+5} = -\frac{d^2}{ds}\frac{-(2s-4)}{(s^2-4s+5)^2}$$

$$= 2\frac{d}{ds}\frac{1(s^2-4s+5)^2-(s-2)\cdot 2(s^2-4s+5)(2s-4)}{(s^2-4s+5)^4}$$

$$= 2\frac{d}{ds}\frac{(s^2-4s+5)\left[(s^2-4s+5)-4(s-2)^2\right]}{(s^2-4s+5)^4}$$

$$\therefore L[t^3e^{2t}\sin t] = \frac{-(-12s+24)(-2s^2+8s-6)}{(s^2-4s+5)^4} = 2\frac{d}{ds}\left(\frac{\left((s^2-4s+5)-4(s^2-4s+4)\right]}{(s^2-4s+5)^3}\right.$$

$$\frac{12(s-2)(-2s^2+8s-6)}{(s^2-4s+5)^4} = 2\frac{d}{ds}\left(\frac{-3s^2+12s-11}{(s^2-4s+5)^3}\right)$$

$$\cancel{\frac{= 12(s-2)\left(s^4+23s^2-8s^3-28s+14\right)}{(s^2-4s+5)^5}}$$

$$= 2\frac{\left((-6s+12)(s^2-4s+5)^3-(-3s^2+12s-11)\cdot 3(s^2-4s+5)^2(2s-4)\right)}{(s^2-4s+5)^6}$$

$$= \frac{2(s^2-4s+5)^2\left[-3(2s-4)(s^2-4s+5)-3(2s-4)(-3s^2+12s-11)\right]}{(s^2-4s+5)^6}$$

$$= \frac{-3\cdot 2(s^2-4s+5)^2(2s-4)\left[(s^2-4s+5)+(-3s^2+12s-11)\right]}{(s^2-4s+5)^6}$$

Ex $L[t^2 e^{-2t}] = \frac{2}{(s+2)^3}$

Ex $L[t^2 \sin 2t] = \frac{4(3s^2-4)}{(s^2+4)^3}$

Ex Find $L\left[\int_0^t t e^{-t} \sin 2t \, dt\right]$

Sol $L[\sin 2t] = \frac{2}{s^2+2^2}$

$L[e^{-t}\sin 2t] = \frac{2}{(s+1)^2+2^2} = \frac{2}{s^2+2s+5}$

$L[t e^{-t}\sin 2t] = (-1)^1 \frac{d^1}{ds^1} L\left[\frac{2}{s^2+2s+5}\right]$

$= - \frac{-2}{(s^2+2s+5)^2}(2s+2)$

$L[t e^{-t}\sin 2t] = \frac{4(s+1)}{(s^2+2s+5)^2}$

Hence $L\left[\int_0^t t e^{-t}\sin 2t \, dt\right] = L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

$L\left[\int_0^t t e^{-t}\sin 2t \, dt\right] = \frac{1}{s} \frac{4(s+1)}{(s^2+2s+5)^2}$

Ex Division by t

If $L[f(t)] = F(s)$ then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]\,ds$

Sol Find $L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = L\left[\frac{f(t)}{t}\right]$

where $f(t) = \frac{e^{-at}-e^{-bt}}{\cancel{t}}$

$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[e^{-at}-e^{-bt}]\,ds$

$= \int_s^\infty L[e^{-at}] - L[e^{-bt}]\,ds$

$= \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$

$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$

$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \left[\log\frac{(s+a)}{s+b}\right]_s^\infty = \log\left(\frac{s+b}{s+a}\right)$

Find $L\left[\frac{\cos at - \cos bt}{t}\right] = L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]\,ds$

where $f(t) = \cos at - \cos bt$

$$= \int_s^\infty \left(L[\cos at - L(\cos bt]\right) ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\left(\log \frac{(s^2+a^2)}{s^2+b^2}\right)_s^\infty$$

$$\left[\frac{\cos at - \cos bt}{t}\right] = \frac{1}{2}\log\frac{(s^2+b^2)}{(s^2+a^2)}$$

Ex find $L\left[\int_0^t \frac{e^t \sin t}{t}\,dt\right]$

Sol $L[\sin t] = 1/s^2+1$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty L[f(t)]\,ds = \int_s^\infty L[\sin t]\,ds = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \left(\operatorname{Tan}^{-1} s\right)_s^\infty = \operatorname{Tan}^{-1}\infty - \operatorname{Tan}^{-1}s = \pi/2 - \operatorname{Tan}^{-1}s = \cot^{-1}s.$$

$$L\left[\frac{e^t \sin t}{t}\right] = \cot^{-1}(s-1)$$

$$L\left[\int_0^t \frac{e^t \sin t}{t}\,dt\right] = \frac{\cot^{-1}(s-1)}{s}$$

Ex Find $L\left[\frac{1-\cos t}{t^2}\right]$

Sol $L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty L[1-\cos t]\,ds$

$$= \int L[1] - L[\cos t]\,ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \left(\log s - \frac{1}{2}\log s^2+1\right)_s^\infty$$

$$= \left(\log s - \log\sqrt{s^2+1}\right)_s^\infty$$

$$L\left[\frac{1-\cos t}{t}\right] = \left(\log\frac{s}{\sqrt{s^2+1}}\right)_s^\infty = \frac{\log\sqrt{s^2+1}}{s}$$

$$L\left[\frac{1-\cos t}{t^2}\right] = L\left[\frac{1}{t}\left(\frac{1-\cos t}{t}\right)\right]$$

$$= \int_s^\infty L\left[\frac{1-\cos t}{t}\right] ds$$

$$= \int_s^\infty \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \left(\log(s^2+1) - \log s^2\right) ds$$

$$= \frac{1}{2}\left[\left(s\log\frac{(s^2+1)}{s^2}\right)_s^\infty + \int_s^\infty \frac{1}{s^2+1} ds\right.$$

$$= \left(\frac{s}{2}\log(1+1/s^2)\right)_s^\infty + \left(\operatorname{Tan}^{-1} s\right)_s^\infty$$

$$= \left(\operatorname{Tan}^{-1}\infty - \operatorname{Tan}^{-1} s\right) - \frac{s}{2}\log(1+1/s^2)$$

$$= \left(\pi/2 - \operatorname{Tan}^{-1} s\right) - \frac{s}{2}\log(1+1/s^2)$$

$$L\left[\frac{1-\cos t}{t^2}\right] = \cot^{-1} s - s/2\log(1+1/s^2) \quad \left(\because \pi/2 - \operatorname{Tan}^{-1} s = \cot^{-1} s\right)$$

① Use Laplace transform of $\int_0^\infty \frac{e^{-t} - e^{-2t}}{t} dt$

② Using Laplace transform, Evaluate $\int_0^\infty \frac{e^{-at}\sin^2 t}{t} dt$

put a = 1 in the above result
Then we have

$$\int_0^\infty e^{-t}\frac{\sin^2 t}{t}\,dt = 1/4\,\log\frac{5}{1} = \frac{1}{4}\log 5.$$

Ex) Evaluate $\int_0^\infty te^{-t}\sin at\,dt = \int_0^\infty t\,e^{-t}\sin t\,dt = 1/2$

Ex) " $\int_0^\infty t\,e^{-4t}\sin 2t\,dt = 11/500$

Laplace transform of some special functions

① Unit step function (Heaviside's unit function)

This function denoted by $u(t-a)$ (or) $H(t-a)$ and is defined as

$$H(t-a) = \begin{cases} 0 & ; t<a \\ 1 & ; t>a \end{cases}$$

Laplace transform of unit step function

P.T $L[H(t-a)] = L[u(t-a)] = \frac{e^{-as}}{s}$

proof

LHS $L[H(t-a)] = \int_0^\infty e^{-st}H(t-a)\,dt$

$$= \int_0^a e^{-st}H(t-a)\,dt + \int_a^\infty e^{-st}H(t-a)\,dt$$

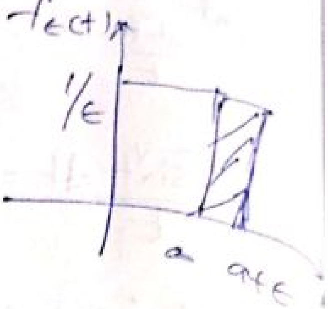
$$= 0 + \int_a^\infty e^{-st}\cdot 1\,dt = \left(\frac{e^{-st}}{-s}\right)_a^\infty$$

$$L[H(t-a)] = 0 - \frac{e^{-as}}{-s} = \frac{e^{-as}}{s}$$

$$\therefore L(H(t-a)) = \frac{e^{-as}}{s}$$

Consider a function

$$f_\epsilon(t-a) = \begin{cases} 1/\epsilon \text{ for } a \le t \le a+\epsilon \\ 0 \;;\; \text{otherwise} \end{cases}$$



Laplace transform

$$L[f_\epsilon(t-a)] = \int_0^\infty e^{-st} f_\epsilon(t-a)\,dt$$

$$= \int_a^{a+\epsilon} e^{-st}\cdot\frac{1}{\epsilon}\,dt + 0$$

$$= \frac{1}{\epsilon}\left(\frac{e^{-st}}{-s}\right)_a^{a+\epsilon}$$

$$= \frac{1}{\epsilon}\left(\frac{e^{-as} - e^{-as-s\epsilon}}{s}\right)$$

$$L[f_\epsilon(t-a)] = e^{-as}\left(\frac{1-e^{-s\epsilon}}{s\epsilon}\right)$$

The limit of $f_\epsilon(t-a)$ as $\epsilon \to 0$ is denoted by $\delta(t-a)$ and is called the Dirac Delta function (or unit impulse function)

Laplace transform of Dirac Delta function

$$L[\delta(t-a)] = \underset{\epsilon\to 0}{Lt}\, L[f_\epsilon(t-a)] = \underset{\epsilon\to 0}{Lt}\, \frac{e^{-as}(1-e^{-s\epsilon})}{s\epsilon} = \left(\frac{0}{0}\right.$$

$$= e^{-as}\, \underset{\epsilon\to 0}{Lt}\, \frac{e^{-s\epsilon}}{s} \quad \text{using L'Hospital rule}$$

$$\therefore\; L[\delta(t-a)] = e^{-as}\cdot 1 = e^{-as}$$

Hence Laplace transform of Dirac Delta function is e^{-as}.

Note $\int_0^\infty \delta(t-a)\,dt = 1$

Ex Find $L[e^{t-3}u(t-3)]$

Sol Since $L[u(t-3)] = \frac{e^{-as}}{s} = \frac{e^{-3s}}{s}$

$$\therefore L\left[e^{t-3}u(t-3)\right] = e^{-3} L\left[e^{t}u(t-3)\right]$$

$$= e^{-3} \cdot \frac{e^{-3(s-1)}}{(s-1)}$$

$$= \frac{e^{-3} \cdot e^{-3s} \cdot e^{3}}{(s-1)} = \frac{e^{-3s}}{(s-1)}$$

$$\therefore L[e^{t-3}u(t-3)] = e^{-3}L\left[e^{t}u(t-3)\right]$$

$$= e^{-3}\cdot\frac{e^{-3(s-1)}}{(s-1)}$$

$$= \frac{e^{-3}\cdot e^{-3s}\cdot e^{3}}{(s-1)} = \frac{e^{-3s}}{(s-1)}$$

Laplace transform of periodic functions

A function f(t) is said to be periodic If and only If $f(t+T)=f(t)$ for some value of T. and for every value of 't'. (where T is non-zero least +ve integer)

If f(t) is a periodic function with periodic T

Then $$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt.$$

Ex Find the Laplace transformation of the rectified semi-wave function defined by

$$f(t) = \sin\omega t, 0 < t < \pi/\omega$$
$$= 0, \pi/\omega < t < \frac{2\pi}{\omega}$$

Sol Since f(t) is a periodic function with period $\frac{2\pi}{\omega}$

$$L[f(t)] = \frac{1}{1-e^{\frac{2\pi s}{\omega}}}\int_0^{\frac{2\pi}{\omega}} e^{-st}f(t)\,dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\int_0^{\pi/\omega} e^{-st}f(t)\,dt + \int_{\pi/\omega}^{\frac{2\pi}{\omega}} e^{-st}f(t)\,dt\right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\int_0^{\pi/\omega} e^{-st}\cdot\sin\omega t\,dt$$

$$= \frac{1}{1-e^{\frac{-2\pi}{\omega}s}}\left[\frac{e^{-st}}{(-s)^2+\omega^2}\left(-s\sin\omega t - \omega\cos\omega t\right)\right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{\frac{-\pi s}{\omega}}}{\omega^2+s^2}\left(-\sin\omega\cdot\frac{\pi}{\omega} - \omega\cos\frac{\pi}{\omega}\cdot\omega\right) - \frac{e^{0}}{s^2+\omega^2}\left(0-\omega\cdot1\right)\right]$$

$$= \frac{\omega}{1-e^{\frac{-2\pi s}{\omega}}}\;\frac{1+e^{-\pi s/\omega}}{s^2+\omega^2}\left(\because \begin{array}{l}\sin 0 = 0\\ \cos 0 = 1\end{array}\right)$$

Ex If $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1; & 1 \le t < 2 \end{cases}$ is a periodic function

with period '2' find it's Laplace transform.

Inverse Laplace transforms

Definition

If $L[f(t)] = F(s)$

Then $f(t) = L^{-1}[F(s)]$

ie If $F(s)$ is the Laplace transform of a function $f(t)$,

Then $f(t)$ is called the inverse Laplace transform of $F(s)$.

Ex ① If $L[e^{at}] = \frac{1}{s-a}$

$\Rightarrow e^{at} = L^{-1}\left[\frac{1}{s-a}\right]$

② If $L[\cos at] = \frac{s}{s^2+a^2}$

$\Rightarrow \cos at = L^{-1}\left[\frac{s}{s^2+a^2}\right]$

properties of inverse Laplace transform

If $F_1(s)$ and $F_2(s)$ are Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively. Then

$$L^{-1}[c_1F_1(s) + c_2F_2(s)] = c_1L^{-1}[F_1(s)] + c_2L^{-1}[F_2(s)]$$

where c_1 & c_2 are constants.

Ex find $L^{-1}\left[\frac{s^2-3s+4}{s^3}\right] = L^{-1}\left[\frac{s^2}{s^3} - \frac{3s}{s^3} + \frac{4}{s^3}\right]$

$= L^{-1}\left[\frac{1}{s}\right] - 3L^{-1}\left[\frac{1}{s^2}\right] + 4L^{-1}\left[\frac{1}{s^3}\right]$

$= 1 - 3t + 4\frac{t^2}{2!}$

$\therefore L^{-1}\left[\frac{s^2-3s+4}{s^3}\right] = 1 - 3t + 2t^2$

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Ex Find $L^{-1}\left[\frac{3(s^2-2)^2}{2s^5}\right]$

Sol $L^{-1}\left[\frac{3(s^2-2)^2}{2s^5}\right] = \frac{3}{2}L^{-1}\left[\frac{(s^2-2)^2}{s^5}\right]$

$= \frac{3}{2}L^{-1}\left[\frac{s^4-4s^2+4}{s^5}\right]$

$= \frac{3}{2}L^{-1}\left[\frac{s^4}{s^5}-\frac{4s^2}{s^5}+\frac{4}{s^5}\right]$

$= \frac{3}{2}L^{-1}\left[\frac{1}{s}-4\frac{1}{s^3}+\frac{4}{s^5}\right]$

$= 3/2 \cdot 1 - 6 \cdot \frac{t^2}{2!} + \frac{3}{2} \cdot 4 \cdot \frac{t^4}{4!}$

$L^{-1}\left[\frac{3(s^2-2)^2}{2s^5}\right] = \frac{3}{2} - 3t^2 + \frac{1}{4}t^4$

First shifting theorem

F

Ex find $L^{-1}\left[\frac{3(s^2-2)^2}{2s^5}\right]$

Sol $L^{-1}\left[\frac{3(s^2-2)^2}{2s^5}\right] = \frac{3}{2}L^{-1}\left[\frac{(s^2-2)^2}{s^5}\right]$

$$= \frac{3}{2}L^{-1}\left[\frac{s^4-4s^2+4}{s^5}\right]$$

$$= \frac{3}{2}L^{-1}\left[\frac{s^4}{s^5}-\frac{4s^2}{s^5}+\frac{4}{s^5}\right]$$

$$= \frac{3}{2}L^{-1}\left[\frac{1}{s}-4\frac{1}{s^3}+\frac{4}{s^5}\right]$$

$$= 3/2\cdot 1-6\cdot\frac{t^2}{2!}+\frac{3}{2}\cdot 4\cdot\frac{t^4}{4!}$$

$$L^{-1}\left[\frac{3(s^2-2)^2}{2s^5}\right] = \frac{3}{2}-3t^2+\frac{1}{4}t^4$$

First shifting theorem

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}(F(s-a)) = e^{at}f(t)$

Ex find $L^{-1}\left[\frac{3s-2}{s^2-4s+20}\right] = L^{-1}\left[\frac{3(s-2)+4}{(s-2)^2+4^2}\right]$

$$= 3L^{-1}\left[\frac{(s-2)}{(s-2)^2+4^2}\right]+4L^{-1}\left[\frac{1}{(s-2)^2+4^2}\right]$$

$$= 3e^{2t}L^{-1}\left[\frac{s}{s^2+4^2}\right]+4L^{-1}\left[\frac{1}{s^2+4^2}\right]e^{2t}$$

$$= 3e^{2t}\cdot\cos 4t+\frac{4}{4}L^{-1}\left[\frac{4}{s^2+4^2}\right]e^{2t}$$

$$\therefore L^{-1}\left[\frac{3s-2}{s^2-4s+20}\right] = 3e^{2t}\cos 4t+e^{2t}\sin 4t.$$

Ex find $L^{-1}\left[\frac{s}{(s+3)^2}\right] = L^{-1}\left[\frac{(s+3)-3}{(s+3)^2}\right]$

$$= e^{-3t}L^{-1}\left[\frac{s-3}{s^2}\right]$$

$$= e^{-3t}\left[L^{-1}\left(\frac{1}{s}\right)-L^{-1}\left(\frac{3}{s^2}\right)\right]$$

$$L^{-1}\left[\frac{s}{(s+3)^2}\right] = e^{-3t}(1-3t)$$

Ex ① find $L^{-1}\left[\frac{1}{s^2+2s+5}\right]$ ④ find $L^{-1}\left[\frac{s+3}{s^2-10s+29}\right]$

② find $L^{-1}\left[\frac{2s+12}{s^2+6s+13}\right]$

③ find $L^{-1}\left(\frac{1}{(s+1)^3}\right]$

Use of partial fractions to find inverse Laplace transform

Ex find $L^{-1}\left[\frac{1}{(s+1)^2(s^2+4)}\right] = L^{-1}[F(s)]$

Sol where $F(s) = \frac{1}{(s+1)^2(s^2+4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{(Cs+D)}{s^2+4}$

$$\Rightarrow \frac{1}{(s+1)^2(s^2+4)} = \frac{A(s+1)(s^2+4)+B(s^2+4)+(Cs+D)(s+1)^2}{(s+1)^2(s^2+4)}$$

$$\Rightarrow 1 = A(s+1)(s^2+4)+B(s^2+4)+(Cs+D)(s+1)^2$$

put $s=-1$

$$1 = 0 + B(1+4) + 0$$

$\Rightarrow 5B = 1 \Rightarrow B = 1/5$

Comparing s^3 coeff. on both sides

$0 = A + C \Rightarrow A = -C$ — ①

Comparing s^2 coeff on both sides

$A + B + 2C + D = 0$ — ②

Comparing s coeff on both sides

$4A + C + 2D = 0$ — ③

Comparing s^0 ie constants on both sides

$4A + 4B + D = 1$ — ④

sub eq ① in ② we get

$-C + B + 2C + D = 0 \Rightarrow B + D + C = 0$

Since $4A + 4B + D = 1$

$-4C + 4B + D = 1$

$$B + D + C = 0$$
$$\underline{\underset{(-)}{4B} + \underset{(-)}{D} \underset{(+)}{-4C} = \underset{(-)}{1}}$$
$$-3B + 5C = -1$$
$$-\frac{3}{5} + 5C = -1$$
$$\Rightarrow 5c = -1 + \frac{3}{5} = \frac{-2}{5}$$
$$\Rightarrow c = -2/25$$

sinc $A = -C = -(-2/25) = \frac{2}{25}$.

sinc $4A + C + 2D = 0$

$$4 \cdot \frac{2}{25} - \frac{2}{25} + 2D = 0$$
$$\Rightarrow +\frac{8}{25} - \frac{2}{25} + 2D = 0 \Rightarrow 2D = -6/25 \Rightarrow D = \frac{-3}{25}$$
$$\Rightarrow \cancel{2}D = \frac{-6}{25} = \frac{-3}{25} \Rightarrow D = \frac{-3}{25} = -\frac{3}{25}$$

$\therefore A = 2/25,\ B = 1/5,\ C = -2/25,\ D = -3/5.$

$$\therefore F(s) = \frac{1}{(s+1)^2(s^2+4)} = \frac{\frac{2}{25}}{s+1} + \frac{\frac{1}{5}}{(s+1)^2} + \frac{-\frac{2}{25}s \mp \frac{3}{25}}{s^2+4}$$

Taking L^{-1} Operator on both sides we get-

$$L^{-1}[F(s)] = L^{-1}\left[\frac{\frac{2}{25}}{s+1} + \frac{\frac{1}{5}}{(s+1)^2} + \frac{1}{25}\frac{(2s \mp 3)}{s^2+4}\right]$$
$$= \frac{2}{25}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{5}L^{-1}\left[\frac{1}{(s+1)^2}\right] - \frac{2}{25}L^{-1}\left[\frac{s}{s^2+4}\right]$$
$$- \frac{3}{25}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$L^{-1}\left[\frac{1}{(s+1)^2(s^2+4)}\right] = e^{-t} \cdot \frac{2}{25} + \frac{e^{-t}t}{5} - \frac{2}{25}\cos 2t - \frac{3}{50}\sin 2t$$

$$L^{-1}\left[\frac{1}{(s+1)^2(s^2+4)}\right] = \frac{e^{-t}}{25}(2+5t) - \frac{2}{25}\cos 2t - \frac{3}{50}\sin 2t.$$

Ex $L^{-1}\left[\frac{s^2}{(s^2+4)(s^2+25)}\right] = \frac{1}{21}(5\sin 5t - 2\sin 2t]$

Ex $L^{-1}\left[\frac{8}{(s^2+1)(s^2+9)(s^2+25)}\right] = \frac{1}{3092}(16\cos t - 24\cos 3t + 8\cos 5t)$

Ex $L^{-1}\left[\frac{2s^2-6s+5}{s^2 \quad 11s-6}\right]$

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Ex Find $L^{-1}\left[\frac{2s+3}{s^3-6s^2+11s-6}\right] = \frac{5}{2}e^{t} - 7e^{2t} + \frac{9}{2}e^{3t}$.

Ex Find $L^{-1}\left[\frac{s}{s^4+4a^4}\right] = \frac{1}{2a^2}\sin at \sinh at$.

Sol

$$L^{-1}\left[\frac{s}{s^4+4a^4}\right] = L^{-1}\left[\frac{s}{(s^2)^2+(2a^2)^2}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2-2as+2a^2)(s^2+2as+2a^2)}\right]$$

$$= \frac{1}{4a}\left[L^{-1}\left[\frac{1}{(s-a)^2+a^2}\right] - L^{-1}\left[\frac{1}{(s+a)^2+a^2}\right]\right]$$

$$= \frac{1}{4a}\left[e^{at}\frac{1}{a}\sin at - e^{-at}\frac{1}{a}\sin at\right]$$

$$= \frac{1}{4a^2}\left(e^{at}-e^{-at}\right)\sin at$$

$$L^{-1}\left[\frac{s}{s^4+4a^4}\right] = \frac{1}{2a^2}\sinh at \sin at$$

Second shifting theorem

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}[e^{-as}F(s)] = G(t)$

where $G(t) = \begin{cases} f(t-a) ; & t>a \\ 0 ; & t<a \end{cases}$

Ex find $L^{-1}\left[\frac{e^{-2s}}{s^2+4s+5}\right]$

Sol $L^{-1}\left[\frac{1}{s^2+4s+5}\right] = L^{-1}\left[\frac{1}{(s+2)^2+1}\right] = e^{-2t}L^{-1}\left[\frac{1}{s^2+1}\right]$

$\therefore L^{-1}\left[\frac{1}{s^2+4s+5}\right] = e^{-2t}\sin t = f(t)$ say

By second shifting theorem

$$L^{-1}\left[\frac{e^{-2s}}{s^2+4s+5}\right] = \begin{cases} e^{-2(t-2)}\sin(t-2) ; & t>2 \\ 0 ; & t<2 \end{cases}$$

(or) $L^{-1}\left[\frac{e^{-2s}}{s^2+4s+5}\right] = e^{-2(t-2)}\sin(t-2)H(t-2)$

where $H(t-2)$ is the Heaviside's unit step function.

Ex Find inverse Laplace transform of $\frac{e^{-\pi(s+2)}}{s+2}$

Ans $e^{-2t} L^{-1}\left[\frac{e^{-\pi s}}{s}\right]$ (using FTT)

$= e^{-2t} u(t-\pi)$ ($L^{-1}\left[\frac{1}{s}\right]=1$] unit step function)

Ex find $L^{-1}\left[\frac{1+e^{-\pi s}}{s^2+1}\right] = \sin t - \sin t\, H(t-\pi)$ ($\because \sin(t-\pi) = -\sin t$)

Change of scale property

If $L[f(t)] = F(s)$ then $L^{-1}[F(as)] = \frac{1}{a} f(t/a)$

Inverse Laplace transform of derivatives

Theorem

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}[F^{(n)}(s)] = (-1)^n t^n f(t)$

where $F^{(n)}(s) = \frac{d^n}{ds^n} F(s)$

Ex Find $L^{-1}\left[\log\frac{(s+1)}{s-1}\right]$

Sol Given $L^{-1}\left[\log\frac{(s+1)}{(s-1)}\right] = L^{-1}[F(s)]$

where $F(s) = \log\frac{(s+1)}{(s-1)} = \log(s+1) - \log(s-1)$

$F'(s) = \frac{1}{s+1} - \frac{1}{s-1}$

Taking L^{-1} on both sides

$L^{-1}[F'(s)] = L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$

$(-1)^1 t^1 f(t) = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s-1}\right]$

$\Rightarrow -t L^{-1}[F(s)] = e^{-t} - e^{t}$

$\Rightarrow L^{-1}\left[\log\frac{(s+1)}{(s-1)}\right] = \frac{e^{t}-e^{-t}}{t} = 2\frac{e^{t}-e^{-t}}{2t} = \frac{2}{t}\sinh t.$

$\therefore L^{-1}\left[\log\frac{(s+1)}{(s-1)}\right] = \frac{2}{t}\sinh t.$

Ex $\log(1+1/s^2)$ Find inverse Laplace transform of

Ex $L^{-1}\left[\cot^{-1}\left(\frac{s+2}{3}\right)\right]$ Ex: $L^{-1}[\cot^{-1}s]$, $L^{-1}\left[\frac{s}{(s^2-25)^2}\right]$

$L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right]$

Ex Find $L^{-1}\left[\frac{s}{(s^2+4)^2}\right]$

Sol $L^{-1}\left[\frac{s}{(s^2+4)^2}\right] = L^{-1}\left[F(s)\right]$

where $F(s) = \frac{1}{(s^2+4)}$

$\Rightarrow$ Diff. w.r.t 's'

$$F'(s) = \frac{-1}{(s^2+4)^2} 2s = \frac{-2s}{(s^2+4)^2}$$

$$\Rightarrow F'(s) = \frac{-2s}{(s^2+4)^2}$$

Taking L^{-1} on both sides

$$L^{-1}[F'(s)] = L^{-1}\left[\frac{-2s}{(s^2+4)^2}\right]$$

$$(-1)^1 t^1 f(t) = -2\, L^{-1}\left[\frac{s}{(s^2+4)^2}\right]$$

$$\Rightarrow \frac{t}{2} f(t) = L^{-1}\left[\frac{s}{(s^2+4)^2}\right]$$

$$\Rightarrow L^{-1}\left[\frac{s}{(s^2+4)^2}\right] = \frac{t}{2} f(t) = \frac{t}{2} L^{-1}[F(s)] = \frac{t}{2} L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{t}{4} \sin 2t$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+4)^2}\right] = \frac{t}{4} \sin 2t.$$

Ex Evaluate $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2a} \sin at$

Inverse laplace transform of integrals

Theorem If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left[\int_s^\infty F(s)\,ds\right] = \frac{f(t)}{t}$

Ex Find $L^{-1}\left[\frac{s+1}{(s^2+2s+2)^2}\right]$

Sol $L^{-1}\left[\frac{s+1}{(s^2+2s+2)^2}\right] = L^{-1}\left[F(s)\right]$

where $F(s) = \frac{s+1}{((s+1)^2+1)^2}$

Taking L^{-1} on both sides

$$L^{-1}[F(s)] = L^{-1}\left[\frac{s+1}{(s+1)^2+1^2}\right] = e^{-t} L^{-1}\left[\frac{s}{s^2+1}\right] = e^{-t} \frac{t \sin t}{2 \cdot 1}$$

Ex $L^{-1}\left[\frac{s+3}{(s^2+6s+13)^2}\right]$

$$= t/4\, e^{-3t} \sin 2t$$

$$L^{-1}\left[\frac{s+1}{(s^2+2s+2)^2}\right]$$

$$= t/2\, e^{-t} \sin t.$$

multiplication by powers of s

Theorem

If $L^{-1}[F(s)] = f(t)$ and $f(0) = 0$ then $L^{-1}[sF(s)] = f'(t)$

Ex Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$

Sol $L^{-1}\left[\frac{s}{(s+2)^2}\right] = L^{-1}\left[s \cdot \frac{1}{(s+2)^2}\right] = L^{-1}\left[s \cdot F(s)\right] = f'(t)$

where $F(s) = \frac{1}{(s+2)^2} \Rightarrow L^{-1}[F(s)] = f(t) = L^{-1}\left[\frac{1}{(s+2)^2}\right]$

$$= \frac{d}{dt}\left(e^{-2t} \cdot t\right) \quad \left(\because L^{-1}\left[\frac{1}{s^2}\right] = t\right)$$

$$= 1 \cdot e^{-2t} - 2e^{-2t} \cdot t$$

$$\therefore L^{-1}\left[\frac{s}{(s+2)^2}\right] = e^{-2t}(1-2t)$$

Find

Ex $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2a}(\sin at + at\cos at)$

Division by s

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)dt$

Ex Find $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]$

Sol $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2+a^2}\right]dt$

$$= \int_0^t \frac{1}{a}\sin at \, dt$$

$$= \frac{-1}{a^2}(\cos at)_0^t$$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1}{a^2}(1-\cos at)$$

$$L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = L^{-1}\left[\frac{1}{s}\,\frac{1}{s(s^2+a^2)}\right] = \int_0^t \frac{1}{a^2}(1-\cos at)dt$$

$$L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = \frac{1}{a^2}\left[t - \frac{\sin at}{a}\right]_0^t = \frac{1}{a^2}\left(t - \frac{\sin at}{a}\right)$$

Convolution

Definition

Let $f(t)$ and $g(t)$ be two functions defined for $t>0$

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

Convolution Theorem

Statement

If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$

Then $L[f(t) \times g(t)] = F(s)G(s)$

(or) $f(t) * g(t) = L^{-1}[F(s)G(s)]$

ie Convolution of two functions $f(t)$ and $g(t)$ is the inverse Laplace transform of their product.

Ex find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}$

Ex $L^{-1}\left[\frac{s^2}{(s^2+4)(s^2+9)}\right] = \frac{1}{5}\left[2\sin 2t - 3\sin 3t\right]$

Ex Solve $\frac{dy}{dt} + 3y + 2\int_0^t y(t)\, dt = t$ by using Laplace transform method

Ex find $L^{-1}\left[\frac{s}{(s^2+1)^2}\right]$

Ex Find $L^{-1}\left[\frac{1}{(s+2)^2(s^2+4)}\right]$

transform of integrals

Theorem

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left[\int_s^{\infty} F(s)\,ds\right] = \frac{f(t)}{t}$

Ex find $L^{-1}\left[\frac{s+1}{(s^2+2s+2)^2}\right]$. $L^{-1}\left[\int_s^{\infty} \cancel{s^2}\frac{1}{s}\left(\frac{s}{s^2+1}\right)ds\right]$

sol $L^{-1}\left[\frac{s+1}{(s^2+2s+2)^2}\right] = L^{-1}\left[\frac{s+1}{((s+1)^2+1)^2}\right]$

$= e^{-t} L^{-1}\left[\frac{s}{(s^2+1)^2}\right]$ ($\because$ F.T.T)

$L^{-1}\left[\frac{s+1}{(s^2+2s+2)^2}\right] = e^{-t}\frac{t}{2}\sin t$

$\left(\because L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2a}\sin at\right)$.

$L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{t}{2\cdot 1}\sin t$

multiplication by powers of s

Theorem

If $L^{-1}[F(s)] = f(t)$ and $f(0)=0$

Then $L^{-1}[sF(s)] = f'(t) = \frac{d}{dt}f(t)$

$L[f'(t)] = sL[f(t)] - f(0)$

$L[f'(t)] = sF(s)$

$f'(t) = L^{-1}[sF(s)]$

Ex $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$

sol $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = L^{-1}\left[s\cdot\frac{s}{(s^2+a^2)^2}\right]$

$= L^{-1}[s\,F(s)]$

$= f'(t) = \frac{d}{dt}f(t)$

$= \frac{d}{dt}L^{-1}[F(s)]$

$= \frac{d}{dt}\left(L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]\right)$

$= \frac{d}{dt}\left(\frac{t}{2a}\sin at\right)$

$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2a}[1.\sin at + ta\cos at]$

Division by s

statement :- If $L^{-1}[F(s)] = f(t)$

Then $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$

$L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$

$\int_0^t f(t)dt = L^{-1}\left[\frac{F(s)}{s}\right]$

Ex $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = L^{-1}\left[\frac{\frac{1}{(s^2+a^2)}}{s}\right]$

$= L^{-1}\left[\frac{F(s)}{s}\right]$

$= \int_0^t L^{-1}[F(s)]dt$ $(f(t) = L^{-1}[F(s)]$

$= \int_0^t L^{-1}\left[\frac{1}{s^2+a^2}\right]dt$

$= \int_0^t \frac{1}{a} L^{-1}\left[\frac{a}{s^2+a^2}\right]dt$

$= \frac{1}{a}\int_0^t \sin at\, dt$

$= \frac{1}{a}\left(\frac{-\cos at}{a}\right)_0^t$

$= \frac{-1}{a^2}(\cos at - \cos 0]$

$= \frac{+1}{a^2}(-\cos at + 1)$

$\therefore L^{-1}\frac{1}{s(s^2+a^2)}] = \frac{1}{a^2}(1-\cos at)$

Ex $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = L^{-1}\left[\frac{1}{s.s(s^2+a^2)}\right] = L^{-1}\left[\frac{F(s)}{s}\right]$

$= \int_0^t L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]dt$

$= \int_0^t \frac{1}{a^2}(1-\cos at)dt$

$= \frac{1}{a^2}\left[t - \frac{\sin at}{a}\right]_0^t$

$L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = \frac{1}{a^2}\left[t - \frac{\sin at}{a}\right]$

Second shifting theorem

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}[e^{-as}F(s)] = G(t)$

where $G(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$

Ex Find $L^{-1}\left[\dfrac{e^{-2s}}{s^2+4s+5}\right]$

Sol $L^{-1}\left[\dfrac{e^{-2s}}{s^2+4s+5}\right] = L^{-1}\left[e^{-2s} \cdot \boxed{\dfrac{1}{s^2+4s+5}}\right]$

$= L^{-1}\left[e^{-as} \boxed{F(s)}\right]$, $a = 2$

$f(t) = L^{-1}[F(s)]$

$= L^{-1}\left[\dfrac{1}{s^2+4s+5}\right]$

$= L^{-1}\left[\dfrac{1}{(s+2)^2+1}\right]$

$= e^{-2t} L^{-1}\left[\dfrac{1}{s^2+1}\right]$

$= e^{-2t} \sin t$

$= f(t)$

By second shifting th

$L^{-1}\left[\dfrac{e^{-2s}}{s^2+4s+5}\right] = G(t) = \begin{cases} e^{-2(t-2)} \sin(t-2) & ; t > 2 \\ 0 & ; t < 2 \end{cases}$

Ex find $L^{-1}\left[\dfrac{1+e^{-\pi s}}{s^2+1}\right]$

Sol. $L^{-1}\left[\dfrac{1+e^{-\pi s}}{s^2+1}\right] = L^{-1}[F(s)]$

$= L^{-1}\left[\dfrac{1}{s^2+1} + \dfrac{e^{-\pi s}}{s^2+1}\right]$

$= L^{-1}\left[\dfrac{1}{s^2+1}\right] + L^{-1}\left[\dfrac{e^{-\pi s}}{s^2+1}\right]$

$L^{-1}[F(s)] = L^{-1}\left[\dfrac{1}{s^2+1}\right]$

$a = \pi$

$f(t) = \sin t$

$= \sin t + \begin{cases} \sin(t-\pi) & ; t > \pi \\ 0 & ; t < \pi \end{cases}$

$\sin(-\theta) = -\sin\theta$

$\sin(t-\pi) = \sin-(\pi-t)$

$= -\sin(\pi-t)$

$= -\sin t$

$\sin(t-\pi) = -\sin t$

$= \sin t + (-\sin t) H(t-\pi)$

$= \sin t - \sin t\, H(t-\pi)$

$L^{-1}\left[\dfrac{1+e^{-\pi s}}{\quad}\right] = \sin t - \sin t\, H(t-\pi)$

Convolution theorem

Def Let f(t), g(t) be two functions defined $t>0$ $\begin{bmatrix}1&2\\3&4\end{bmatrix}$

$$f(t) * g(t) = \int_0^t f(u)\,g(t-u)\,du$$

Convolution theorem

St If $L[f(t)] = F(s)$, $L[g(t)] = G(s)$

Then $L[f(t) * g(t)] = F(s)G(s)$

ie convolution of Laplace transform of f(t) & g(t) is the product of their Laplace transforms

$$L[f(t) * g(t)] = F(s)\,G(s)$$

$$\Rightarrow \boxed{f(t) * g(t) = L^{-1}[F(s)\,G(s)]}$$

Ex Find $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ by using convolution theorem.

Sol

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right]$$

$$= L^{-1}[F(s)\,G(s)]$$

By convolution theorem

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \underset{f}{\cos at} * \underset{g}{\cos bt}$$

By using convolution Def

$$= \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t \cos au \cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t 2\,\underset{A}{\cos au}\,\underset{B}{\cos b(t-u)}\,du$$

$$= \frac{1}{2}\int_0^t \left[\cos(au+bt-bu) + \cos(au-bt+bu)\right] du$$

$(\because \cos(A+B)+\cos(A-B) = 2\cos A\cos B)$

$$= \frac{1}{2}\left[\frac{\sin((a-b)u+bt)}{a-b} + \sin((a+b)u-bt)\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin((a-\cancel{b})t \cancel{+bt})}{a-b} + \frac{\sin((a+\cancel{b})t \cancel{-bt})}{a+b} - \left(\frac{\sin(bt)}{a-b} + \frac{\sin(-bt)}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{\sin at}{a-b} + \frac{\sin at}{a+b}\right) - \left(\frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{\cancel{a+b}+a-\cancel{b}}{a^2-b^2}\right)\sin at - \left(\frac{\cancel{a}+b-(\cancel{a}-b)}{a^2-b^2}\right)\sin bt\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at}{a^2-b^2} - \frac{2b\sin bt}{a^2-b^2}\right]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \left(\frac{a\sin at - b\sin bt}{a^2-b^2}\right)$$

Ex $L^{-1}\left[\frac{s^2}{(\underset{2^2}{s^2+4})(\underset{3^2}{s^2+9})}\right] = \frac{2\sin 2t - 3\sin 3t}{4-9} = \frac{3\sin 3t - 2\sin 2t}{5}$

Ex Find $L^{-1}\left[\frac{s}{(s^2+1)^2}\right]$ by using convolution th

Sol $L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = L^{-1}\left[\frac{s}{(s^2+1)}\cdot\frac{1}{(s^2+1)}\right]$

$$= L^{-1}\left[F(s)\,G(s)\right] = f(t) * g(t)$$

$$= L^{-1}\left[\frac{s}{s^2+1}\right] * L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \underset{f}{\cos t} * \underset{g}{\sin t} = f(t) * g(t)$$

$$= \int_0^t f(u)\,g(t-u)\,du$$

$$= \int^t \cos u \sin(t-u)\,du$$

$$= \int_0^t \cos u \sin(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \underline{\underline{2\cos \underset{A}{u} \sin(\underset{B}{t-u})}}\,du$$

$$\left(\because \sin(A+B) - \sin(A-B) = 2\underline{\underline{\cos A \sin B}}\right)$$

$$= \frac{1}{2}\int_0^t \left[\sin(\cancel{u}+t-\cancel{u}) - \sin(u-(t-u))\right] du$$

$$= \frac{1}{2}\int_0^t \left[\sin t - \sin(\boxed{2u}-t)\right] du$$

$$= \frac{1}{2}\left[\sin t\,(u)_{u=0}^{t} + \left(\frac{\cos(2u-t)}{2}\right)_{u=0}^{t}\right]$$

$$= \frac{1}{2}\left[\sin t\,(t-0)\right] + \frac{\cos t}{2} - \frac{\cos(-t)}{2}\Big]$$

$$= \frac{1}{2}\left[t\sin t + \cancel{\frac{\cos t}{2}} - \cancel{\frac{\cos t}{2}}\right]$$

$$\Rightarrow \boxed{\frac{t}{2}\sin t} = L^{-1}\left[\frac{s}{(s^2+\boxed{1})^2}\right]$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2a}\sin at ✓$$

$$L^{-1}\left[\frac{s}{(s^2+3^2)^2}\right] = \boxed{\frac{t}{2\cdot 3}\sin 3t}$$

Ex solve $\frac{dy}{dt} + 3y + 2\int_0^t \underline{y(t)}\,dt = t$, ~~$y(0)=$~~

using Laplace transform.

Sol Given $y'(t) + 3y(t) + 2\int_0^t y(t)\,dt = t$

Taking Laplace transform on soln

$$L\left[y'(t) + 3y(t) + 2\int_0^t y(t)\,dt\right] = L[t]$$

$$L[y'(t)] + 3L[y(t)] + 2L\left[\int_0^t \underline{\underline{y(t)}}\,dt\right] = L[t]$$

$\Rightarrow$

$$\Rightarrow L[y(t)] = \frac{1}{s(s^2+3s+2)} + \frac{as}{s^2+3s+2}$$

Taking inverse Laplace transform

$$y(t) = \underbrace{L^{-1}\left[\frac{1}{s(s^2+3s+2)}\right]}_{Ⓐ} + a\underbrace{L^{-1}\left[\frac{s}{s^2+3s+2}\right]}_{Ⓑ} \text{ — ①}$$

Consider $L^{-1}\left[\frac{1}{s(s^2+3s+2)}\right] = L^{-1}[F(s)]$

$$\therefore F(s) = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+2)} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

$$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

put $s = 0$

$$1 = 2A \Rightarrow A = 1/2$$

put $s = -1$

$$1 = B(-1)1 \Rightarrow -B = 1 \Rightarrow B = -1$$

put $s = -2$

$$1 = C(-2)(-1) \Rightarrow 2C = 1 \Rightarrow C = 1/2$$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = L^{-1}\left[\frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}\right]$$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2}\cdot 1 - e^{-t} + \frac{1}{2}e^{-2t} \text{ — Ⓐ}$$

Consider $L^{-1}\left[\frac{s}{s^2+3s+2}\right] = L^{-1}[F(s)]$

$$\therefore F(s) = \frac{s}{s^2+3s+2} = \frac{s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{-1}{s+1} + \frac{2}{s+2}$$

$$s = A(s+2) + B(s+1)$$

put $s = -1$

$$-1 = A \Rightarrow A = -1$$

put $s = -2$

$$-2 = -B \Rightarrow B = 2$$

Taking L^{-1} on both side

$$L^{-1}\left[\frac{s}{s^2+3s+2}\right] = L^{-1}\left[\frac{-1}{s+1}\right] + 2L^{-1}\left[\frac{1}{s+2}\right]$$

$${}^{-2t} \text{ Ⓑ}$$

put $s=-1$.

$$1 = B(-1)1 \Rightarrow -B=1 \Rightarrow B=-1$$

put $s=-2$

$$1 = C(-2)(-1) \Rightarrow 2C=1 \Rightarrow C=1/2$$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = L^{-1}\left[\frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}\right]$$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2}\cdot 1 - e^{-t} + \frac{1}{2}e^{-2t} \text{ — } (A)$$

Consider $L^{-1}\left[\frac{s}{s^2+3s+2}\right] = L^{-1}[F(s)]$

$$\therefore F(s) = \frac{s}{s^2+3s+2} = \frac{s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{-1}{s+1} + \frac{2}{s+2}$$

$$s = A(s+2) + B(s+1)$$

put $s=-1$

$$-1 = A \Rightarrow A=-1$$

put $s=-2$

$$-2 = -B \Rightarrow B=2$$

Taking L^{-1} on both side

$$L^{-1}\left[\frac{s}{s^2+3s+2}\right] = L^{-1}\left[\frac{-1}{s+1}\right] + 2L^{-1}\left[\frac{1}{s+2}\right]$$

$$= -e^{-t} + 2e^{-2t} \text{ — } (B)$$

sub (A) & (B) in eq (1)

$$y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} + a\left(2e^{-2t} - e^{-t}\right)$$

Application of Laplace transform to solve ordinary diff. equations

Procedure

① Take the Laplace transform on both sides

② Use formulas, apply initial conditions

③ Rearrangement of terms

④ Take L^{-1} operator on both sides and get the sol of O.D.E

Ex Using Laplace transform, solve the DE

$$\frac{d^2y}{dt^2} + \frac{2dy}{dt} + 5y = e^{-t}\sin t, \text{ Given that } y(0)=0,\ y'(0)=$$

Sol $y''(t) + 2y'(t) + 5y(t) = e^{-t}\sin t$

step-1 $L[y'' + 2y' + 5y] = L[e^{-t}\sin t]$

$L[y''(t)] + 2L[y'(t)] + 5L[y(t)] = L[e^{-t}\sin t]$

step-2 $(s^2L[y(t)] - sy(0) - y'(0)) + 2[sL[y(t)] - y(0)] + 5L[y(t)] = \frac{1}{(s+1)^2+1}$

Given $y(0)=0$, $y'(0)=1$

$(s^2L[y(t)] - s.0 - 1) + 2(sL[y(t)] - 0) + 5L[y(t)] = \frac{1}{s^2+2s+2}$

$L[y(t)][s^2+2s+5] - 1 = \frac{1}{s^2+2s+2}$

$\Rightarrow L[y(t)](s^2+2s+5) = \frac{1}{s^2+2s+2} + 1$

$\Rightarrow L[y(t)] = \frac{1}{(s^2+2s+2)(s^2+2s+5)} + \frac{1}{s^2+2s+5}$ —①

Consider

$$\frac{1}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$\Rightarrow 1 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)$

C.C of s^3 on both sides

$A + C = 0$ —②

C.C of s^2 on both sides

$2A + B + 2C + D = 0$ —③

C.C of s on both sides

$5A + 2B + 2C + 2D = 0$ —④

$5B + 2D = 1$ ⑤

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Application of Laplace transform to solve ordinary diff. equations

working rule

① Take the Laplace transform on both sides

② Use formulas, apply initial conditions

③ Rearrangement of terms

④ Take L^{-1} operator on both sides and get the sol of O.D.E

Ex Using Laplace transform, solve the DE

$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$, Given that $y(0)=0$, $y'(0)=1$

Sol $y''(t)+2y'(t)+5y(t) = e^{-t}\sin t$

step-1 $L[y''+2y''+5y] = L[e^{-t}\sin t]$

$L[y''(t)]+2L[y'(t)]+5L[y(t)] = L\{e^{-t}\sin t]$

step-2 $(s^2L[y(t)] - sy(0) - y'(0)) + 2[sL[y(t)] - y(0)]$

$+ 5L[y(t)] = \frac{1}{(s+1)^2+1}$

Given $y(0)=0$, $y'(0)=1$

$((s^2L[y(t)] - s.0 - 1) + 2(sL[y(t)] - 0)$

$+ 5L[y(t)] = \frac{1}{s^2+2s+2}$

$L[y(t)][s^2+2s+5] - 1 = \frac{1}{s^2+2s+2}$

$\Rightarrow L[y(t)](s^2+2s+5] = \frac{1}{s^2+2s+2} + 1$

$\Rightarrow L[y(t)] = \frac{1}{(s^2+2s+2)(s^2+2s+5)} + \frac{1}{s^2+2s+5}$ — ①

Consider

$\frac{1}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$

$\Rightarrow 1 = (As+B)(s^2+2s+5) + ((Cs+D)(s^2+2s+2)$

C.C of s^3 on both sides

$A + C = 0$ — ②

C.C of s^2 on both sides

$2A+B+2C+D = 0$ — ③

C.C of s on both sides

$5A+2B+2C+2D = 0$ ④

$5B+2D=1$ ⑤

$A + C = 0$ — ②

$2A + B + 2C + D = 0$ — ③

$5A + 2B + 2C + 2D = 0$ — ④

$5B + 2D = 1$ — ⑤

From ② $A + C = 0 \Rightarrow A = -C$

Sub in ③

$\cancel{2A} + B - 2\cancel{A} + D = 0$

$\Rightarrow \underline{B + D = 0}$ — ⑥

Solving ⑤ & ⑥

$5B + 2\cancel{D} = 1$

$\underline{2B + \cancel{2D} = 0}$
$(-) \quad (-) \quad (-)$

$\underline{3B = 1}$

$\Rightarrow \boxed{B = 1/3}$

Since ⑥ eq.

$B + D = 0$

$\Rightarrow D + 1/3 = 0 \Rightarrow \boxed{D = -1/3}$

From ④

$5A + 2B + 2C + 2D = 0$

$5A + \cancel{2/3} - 2A + \cancel{2/3} = 0$

$5A \mp 2/3 - 2A$

$\Rightarrow 3A = 0 \Rightarrow A = 0$

From ② $A = -C \Rightarrow C = -A = 0$

$\therefore \boxed{A = C = 0}$

$B = +1/3, \; D = -1/3$

$\therefore A = C = 0,$

Sub. all these in eq ① we get

$$\frac{1}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$$\frac{1}{(s^2+2s+2)(s^2+2s+5)} = \frac{+1/3}{s^2+2s+2} \,\bar{\times}\, \frac{\frac{1}{3}}{s^2+2s+5}$$

$$L[y(t)] = \frac{+\frac{1}{3}}{s^2+2s+2} + \frac{-\frac{1}{3}}{s^2+2s+5} + \frac{1}{s^2+2s+5}$$

$+1/3$

$$y(t) = L^{-1}\left[\frac{-1}{3}\frac{1}{s^2+2s+2} + \frac{2}{3}\frac{1}{s^2+2s+5}\right]$$

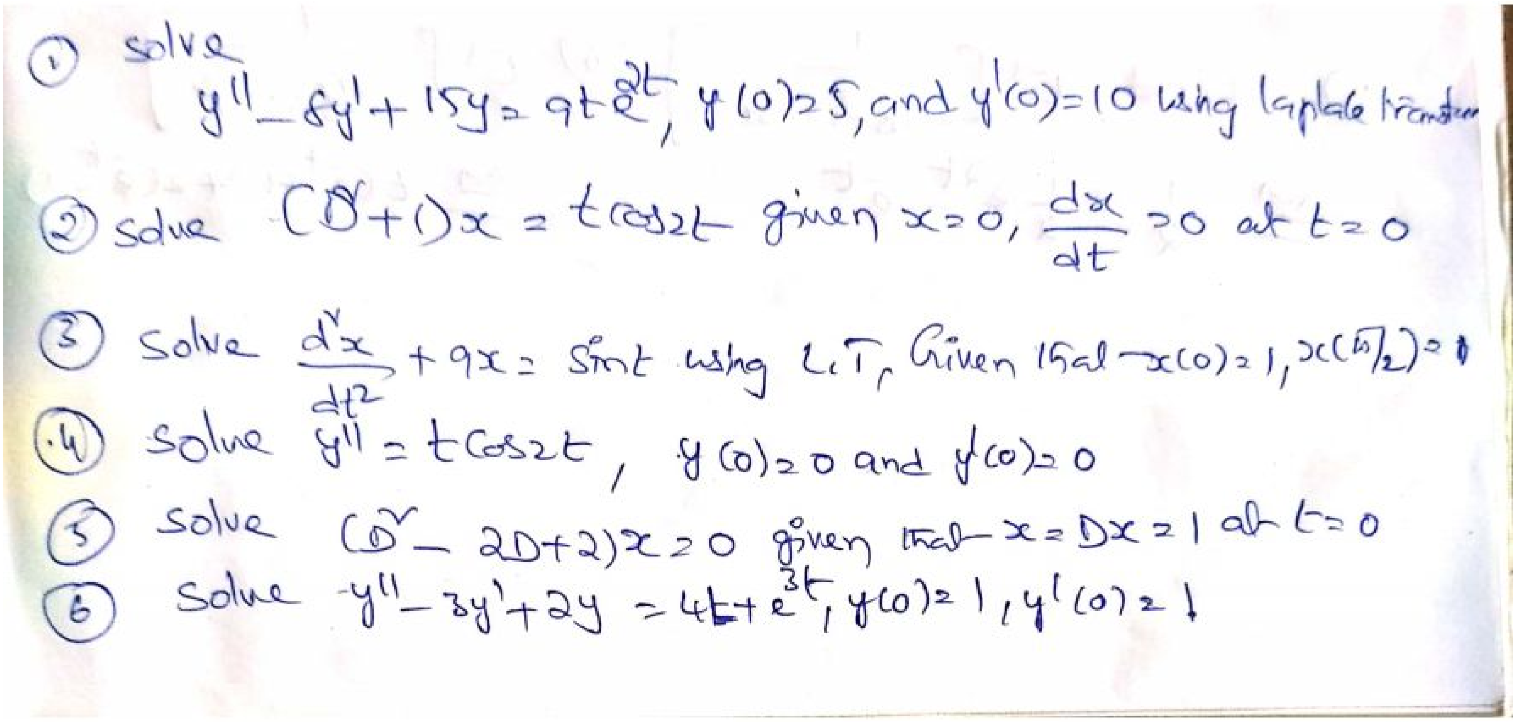
$$= \frac{-1}{3}L^{-1}\left[\frac{1}{s^2+2s+2}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s^2+2s+5}\right] \text{ (by lp)}$$

$$= \frac{-1}{3}L^{-1}\left[\frac{1}{(s+1)^2+1}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= \frac{-1}{3}e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{\cancel{2}\cdot 1\, e^{-t}}{3\cdot\cancel{2}}L^{-1}\left[\frac{2(1)}{s^2+(2)^2}\right]$$

$$= \frac{-1}{3}e^{-t}\sin t + \frac{1}{3}e^{-t}\sin 2t$$

$$\therefore \boxed{y(t)} = \frac{e^{-t}}{3}(\sin 2t - \sin t).$$



① solve
$y'' - 8y' + 15y = 9te^{2t}$, $y(0) = 5$, and $y'(0) = 10$ using laplace transform

② solve $(D^2 + 1)x = t\cos 2t$ given $x = 0$, $\frac{dx}{dt} = 0$ at $t = 0$

③ solve $\frac{d^2x}{dt^2} + 9x = \sin t$ using L.T, Given that $x(0) = 1$, $x(\pi/2) = 1$

④ solve $y'' = t\cos 2t$, $y(0) = 0$ and $y'(0) = 0$

⑤ solve $(D^2 - 2D + 2)x = 0$ given that $x = Dx = 1$ at $t = 0$

⑥ solve $y'' - 3y' + 2y = 4t + e^{3t}$, $y(0) = 1$, $y'(0) = 1$

Solutions

① $y(t) = 4e^{2t} + 3te^{2t} + 3e^{3t} - 2e^{5t}$

② $x(t) = \frac{4}{9}\sin 2t - \frac{5}{9}\sin t - \frac{t\cos 2t}{3}$

③ $x(t) = \frac{1}{8}\left(\sin t - \frac{1}{3}\sin 3t\right) + \cos 3t - \frac{5}{6}\sin 3t$

④ $y(t) = \frac{1}{4}\left(\sin 2t - t - t\cos 2t\right)$

⑤ $x = e^{t}\cos t$

⑥ $y = 3 + 2t - \frac{5}{2}e^{t} + \frac{1}{2}e^{3t}.$

Convolution Theorem

① Using Convolution theorem find $L^{-1}\left[\frac{1}{(s^2+9)(s+1)^2}\right]$

Ans $-\frac{1}{50}\left[\cos 3t + \frac{4}{3}\sin 3t\right] + \frac{e^{-t}}{50} + \frac{te^{-t}}{10}$

② Find $L^{-1}\left[\frac{1}{(s+2)^2(s^2+4)}\right]$

Ans

③ Solve integro differential equation

$$y(t) = 1 - e^{-t} + \int_0^t y(t-u)\sin u\,du$$

Ans $1 - e^{t} + L^{-1}\left[\frac{1}{s^3(s+1)}\right]$

④ Find $L^{-1}\left[\frac{1}{(s^2+4)(s^2+25)}\right] = \frac{1}{21}\left[5\sin 5t - 2\sin 2t\right]$

⑤ Find $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$ ⑥ $L^{-1}\left[\frac{1}{s^2(s+1)^2}\right]$

Ans $\frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}$

Ans $t(e^{-t}+1) + 2(e^{-t}-1)$

Unit-III Fourier Series and Fourier transforms

Section-A Fourier series

- * periodic functions
- * Fourier Series of periodic (II) functions
- * Dirichlet's conditions
- * Even and odd functions — (IV)
- * Change of Interval — (V)
- * Half range sine & cosine series — (VI)

Section-B Fourier transforms

- * Fourier integral theorem (without proof)
- * Fourier sine and cosine Integrals
- * Sine & cosine transforms
- * properties
- * Inverse transforms
- * Finite Fourier transforms

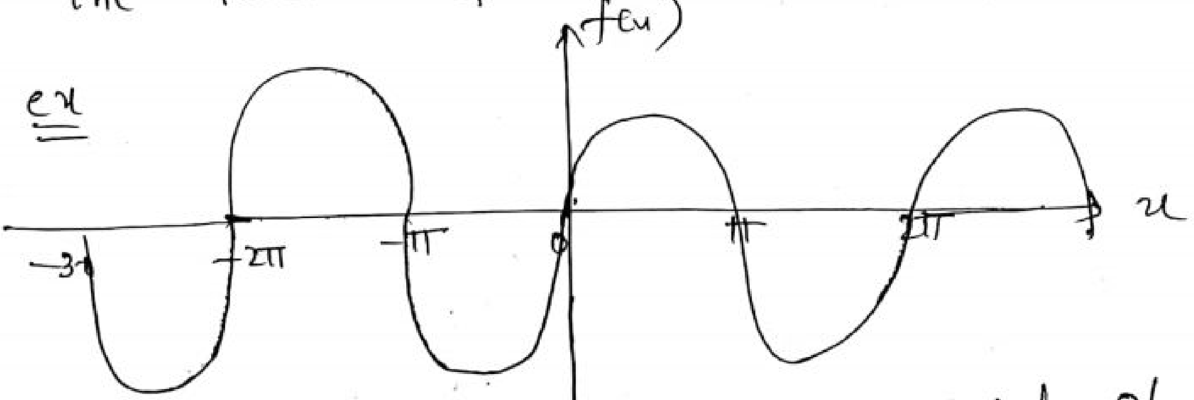
Section-A

periodic function : A function $f(x)$ is said to be of period T if for all real value of x, $f(x+T) = f(x)$

$$f(x) = f(x+T) = f(x+2T) = f(x+3T)$$

Since $\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \sin(x+6\pi) =$ the function of $\sin x$ is periodic with 2π.

ex



In a similar manner the period of $\cos x$ is 2π

The period of $\tan x$ is π

$\sin 3x = \sin(2\pi + 3x) = \sin 3\left(\frac{2\pi}{3} + x\right)$

period $2\pi/3$

TOPIC-II Fourier series of period function

The Fourier Series of the function $f(x)$ in the interval $c \le x \le c+2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\,dx$

a_n ~~b_n~~ $= \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\cos nx\,dx$

$b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\sin nx\,dx$

These values a_0, a_n, b_n are known as Euler's formulae

②

Topic-III Dirichlets conditions

** conditions for fourier Expansion

Dirichlet has formulated certain conditions known as Dirichlet conditions under which certain functions valid Fourier Expansions

A fuction f(x) has a valid Fourier Series Expansion of the form

$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where a_0, a_n, b_n are constants provided single valued

1) f(x) is well defined, peridic, and finite

ii) f(x) has finite number of finite discontinuities in any one period

iii) f(x) has atmost a finite number of maxima and minima in the Interval of definition

The above conditions are sufficient but not necessary

Topic-IV Even and odd functions

Even function: A function f(x) is said to be even function if $f(-x) = f(x)$

and odd $f(-x) = -f(x)$

ex: x^2, x^4+x^2+1, cos x sec x → even fns

x x^3, x^5+2x^3+3, sin x, cosec tan x → odd fns problems from TOPIC – II

TOPIC – IV

TOPIC – V

TOPIC – VI

TOPIC – V Change of Interval

functions ~~of~~ in the Interval $(0, l)$ $(0, 2l)$ other than $(-\pi\ \pi)$ $(0, 2\pi)$

In engineering problems the period of the function to be expanded is not 2π but some other quantity say $2l$ functions of period $2l$. this Interval must be converted to the length 2π

TOPIC – VI Half Range Fourier Series $[0, \pi]$

1) Sine Series

2) Cosine Series

we can Express $f(x)$ as Sine Series as well as Cosine Series

Some Important formulaes

1) $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases}$

2) $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & m = n = 0 \\ \pi & m = n > 0 \end{cases}$

3) $\sin n\pi = 0$ (ie $\sin(\text{any } \pi) = 0$

$\sin 1\pi = \sin 2\pi = \sin 3\pi = \sin 4\pi \cdots = 0$

4) $\cos n\pi = (-1)^n \quad n \in Z$

$\cos 1\pi = -1$

$\cos 3\pi = -1$

$\cos 5\pi = -1$

$\cos(\text{odd } \pi) = -1$

$\cos 2\pi = 1$

$\cos 4\pi = 1$

$\cos 6\pi = 1$

$\cos(\text{even } \pi) = 1$

5) $\sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$

$\sin\frac{\pi}{2} \quad \sin\frac{3\pi}{2} \quad \sin\frac{5\pi}{2} \quad \sin\frac{7\pi}{2}$

$\sin 9\pi/2 = (-1)^n$

⑥ $\cos\left(n + \frac{1}{2}\right)\pi = 0$

~~sin~~ 1 $\cos\frac{\pi}{2} \quad \cos 3\pi/2$

$\cos 5\pi/2 \quad \cos 9\pi/2 = 0$

7) $\sin\frac{n\pi}{2} = \begin{cases} (-1)^{\frac{n-1}{2}} & n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

8) $\cos\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2} & \text{if } n \text{ is even} \end{cases}$

Ⓐ

Formula's on Topic-II

Case I :- function is continuous in $(0, 2\pi)$

then $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$\boxed{0 < x < 2\pi}$

$$\boxed{\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \end{aligned}}$$

Case-II :- function is discontinuous function

(a) $0 < x < \pi$
$\pi < x < 2\pi$
$[0, 2\pi]$

(b) $-\pi < x < 0$
$0 < x < \pi$
$[-\pi, \pi]$

$$\boxed{\begin{aligned} a_0 &= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right. \\ a_n &= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos nx\,dx + \int_{\pi}^{2\pi} f(x)\cos nx\,dx\right. \\ b_n &= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\sin nx\,dx + \int_{\pi}^{2\pi} f(x)\sin nx\,dx\right. \end{aligned}}$$

(b)

$$\boxed{\begin{aligned} a_0 &= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right] \\ a_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_0^{\pi} f(x)\cos nx\,dx\right. \\ b_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_0^{\pi} f(x)\sin nx\,dx\right. \end{aligned}}$$

④

Formulas from TOPIC IV   Ⓑ

we check the Interval first whether it's $[-\pi \; \pi]$ continuous function or not then proceed

If f(x) is even fn → we find a_0, a_n

If f(x) is odd fn → we find b_n

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx$ $\boxed{f(x) \text{ is even}}$	$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\,dx$ f(x) is odd fn

$f(x) = \cos x$ even fn

we find a_0, a_n

$f(x) = x^2$ odd fn

we find b_n only.

Formula's from TOPIC – VI → Ⓒ

$\boxed{0 < x < \pi}$

Case-I Half range cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx$$

Half range sine series

$$f(x) = \cancel{\frac{a_0}{2}} \cancel{+} \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$\boxed{\text{Formula's from Topic IV}}$ — Ⓓ

f(x) defined b/w $[0, 2l]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\boxed{\begin{aligned} a_0 &= \frac{1}{l}\int_0^{2l} f(x)\,dx \\ a_n &= \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \\ b_n &= \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \end{aligned}}$$

$[0, l]$ ~~$f(x) = \frac{a_0}{2} + \sum_{n=1}$~~

Half range cosine series

$$\boxed{a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx \qquad a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx}$$

Half range sine series $[0, l]$

$$\boxed{b_n = \frac{2}{l}\int_0^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)}$$

then Half range cosine series

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)}$$

Half range sine series

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)}$$

problems on Topic-II

⑤

1. Find the Fourier series representing $f(x) = x, \; 0 < x < 2\pi$. sketch the graph of $f(x)$ from -4π to 4π.

Sol) Let the function $f(x) = x$ be represented by the fourier series.

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots\ldots ①$$

Then $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$

and $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1.\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left(\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\right)_0^{2\pi}$$

$$= \frac{1}{\pi}\left(\frac{1}{n^2}\cos 2n\pi - \frac{1}{n^2}\right) = \frac{1}{\pi}\left(\frac{1}{n^2} - \frac{1}{n^2}\right)$$

$$= 0 \quad [\because \cos 2n\pi = \cos 0 = 1]$$

Finally $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

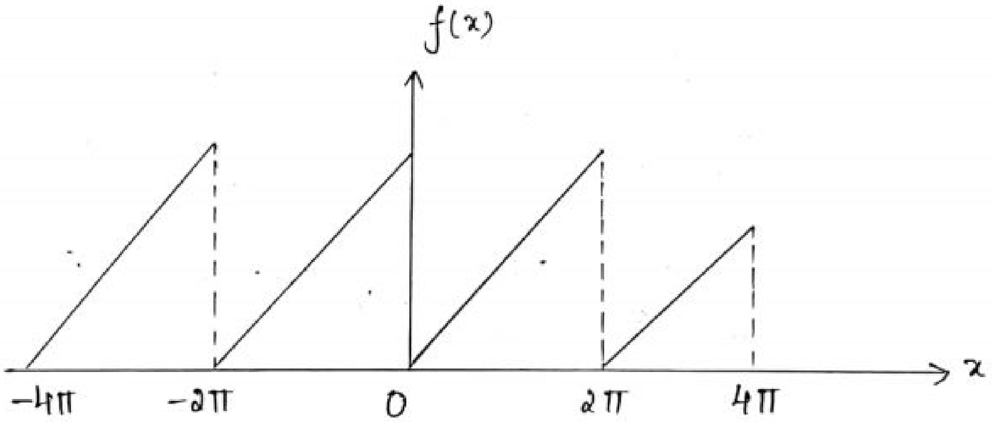
$$= \frac{1}{\pi}\left[\frac{-1}{n} x\cos nx + \frac{1}{n^2}\sin nx\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{-1}{n}2\pi\cos 2n\pi + 0\right) - (0+0)\right] = \frac{-2}{n} \quad (\because \cos 2n\pi = 1)$$

Substituting the values of a_0, a_n and b_n in (1), we get

$$x = \pi - 2\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$= \pi - 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right) \cdots ②$$



Graph of $f(x) = x$ in $|-4\pi, 4\pi|$

2. Expand $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 < x < 2\pi$ in a fourier series.

(or) Obtain the fourier series to represent $f(x) = \frac{1}{4}(\pi - x^2)$ in $0 < x < 2\pi$.

(or) If $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $(0, 2\pi)$, show that $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$. Hence obtain $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$.

Sol) Given $f(x) = \left[\frac{\pi - x}{2}\right]^2 = \frac{(\pi - x)^2}{4}$ in $(0, 2\pi)$. ⑥

The fourier series of $f(x)$ in $[0, 2\pi]$ is given by

$$\frac{(\pi - x)^2}{4} \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a_n, b_n are obtained through Euler's formulae.

Then $a_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4} dx = \frac{1}{4\pi}\left[-\frac{1}{3}(\pi - x)^3\right]_0^{2\pi}$

$$= \frac{-1}{12\pi}\left[(-\pi^3) - \pi^3\right] = \frac{\pi^2}{6}$$

and $a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4} \cos nx \, dx$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}, \text{ by parts}$$

$$= \frac{1}{4\pi}\left[\left(0 + \frac{2\pi \cos 2n\pi}{n^2} + 0\right) - \left(0 - \frac{2\pi \cos 0}{n^2} + 0\right)\right]$$

$$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] \quad [\because \cos 2n\pi = \cos 0 = 1]$$

$$= \frac{1}{4\pi}\left[\frac{4\pi}{n^2}\right] = \frac{1}{n^2} \quad (\text{if } n \neq 0).$$

Finally $b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4} \sin nx \, dx$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right) - 2(\pi-x)(-1)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2\cos 2n\pi}{n} - 0 + 2\,\frac{\cos 2n\pi}{n^3}\right) - \left(\frac{-\pi^3}{n} - 0 + \frac{2}{n^3}\right)\right]$$

$$= \frac{1}{4\pi}\left[\left(\frac{-\pi^2}{n} + \frac{2}{n^3}\right) - \left(\frac{-\pi^2}{n} + \frac{2}{n^3}\right)\right] \quad (\because \cos 2n\pi = 1)$$

Substituting the values of a_0, a_n and b_n in (1), we get

$$\left(\frac{\pi-x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx = \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots \quad ②$$

which is the required fourier series.

Deduction : Putting $x = \pi$ in ②, we obtain

$$0 = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} \cancel{+} - \frac{1}{3^2} + \cdots = \frac{\pi^2}{12} - \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\cdot\right)$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

3. Obtain the fourier series for the function $f(x) = x\sin x$, $0 < x < 2\pi$.

Sol) Let $x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx \cdots$ ①

Then $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$

$$= \frac{1}{\pi}\left[x(-\cos x) - 1(-\sin x)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x\cos x + \sin x\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(-2\pi+0)-(0+0)\right] = -2$$

and $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x\cos nx)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x-\sin(n-1)x\right]dx$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right\} - 1.\left\{-\frac{\sin(n+1)x}{(n+1)^2}+\frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}, (n\neq 1)$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1}+\frac{\cos 2(n-1)\pi}{n-1}\right\}\right], (n\neq 1)$$

$$= -\frac{1}{n+1}+\frac{1}{n-1} = \frac{2}{n^2-1}, \quad (n\neq 1)$$

If $n=1$, we have.

$$a_1 = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx = \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1.\left(\frac{-\sin 2x}{4}\right)\right]_0^{-2\pi}$$

$$= \frac{1}{2\pi}(-\pi) = -\frac{1}{2}$$

Finally $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin \sin nx\,dx$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x\sin nx)\,dx \ldots\ldots ②$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x\right]dx$$

$[\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$

$$\therefore b_n = \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - 1.\left\{\frac{-\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}, (n\neq 1)$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right], (n\neq 1)$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right], (n\neq 1)$$

$\therefore b_n = 0$ for $n \neq 1$

If $n=1$, then

$$b_1 = \frac{1}{2\pi}\int_0^{2\pi} x.\, 2\sin^2 x\, dx \quad [\text{Putting } n=1 \text{ in } ②]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\,dx \quad [\text{Integration by parts}]$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1.\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\cdot 2\pi - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \pi$$

Substituting the values of a_0, a_n and b_n in ①, we get

$$x\sin x = -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx + \pi\sin x$$

$$= -1 + \pi\sin x - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{\cos nx}{n^2-1}$$

This is the required Fourier Series

⑧

4. Find the fourier expansion of $f(x) = x\cos x ; 0 < x < 2\pi$

Sol) The fourier series of $f(x) = x\cos x$, is given by

$$f(x) = x\cos x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ --- } ①$$

Here $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{2\pi} x\cos x \, dx$

$$= \frac{1}{\pi}\left[x \cdot \sin x - 1 \cdot (-\cos x)\right]_0^{2\pi} \text{, by parts}$$

$$= \frac{1}{\pi}(x\sin x + \cos x)_0^{2\pi}$$

$$= \frac{1}{\pi}[0 + \cos 2\pi) - (0+1)] = \frac{1}{\pi}(1-1) = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos x \cos nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\cos x \cos nx)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\cos(1+n)x + \cos(1-n)x]\, dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{\sin(1+n)x}{1+n} + \frac{\sin(1-n)x}{1-n}\right) - 1 \cdot \left(\frac{-\cos(1+n)x}{(1+n)^2} - \frac{\cos(1-n)x}{(1-n)^2}\right)\right]_0^{2\pi} \quad (n \neq 1).$$

$$= \frac{1}{2\pi}\left[\frac{\cos(1+n)x}{(1+n)^2} + \frac{\cos(1-n)x}{(1-n)^2}\right]_0^{2\pi} \quad [\because \sin(1+n)2\pi = \sin(1-n)2\pi = 0]$$

$$\therefore a_n = 0, (n \neq 1) \quad [\because \cos 2(n+1)\pi = \cos 2(n-1)\pi = 1]$$

If $n=1$, $$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\cos^2 x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x(1+\cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[x\left(x+\frac{\sin 2x}{2}\right) - 1\cdot\left(\frac{x^2}{2} - \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(2\pi^2 + 0 + \frac{1}{4}\right) - \left(0+0+\frac{1}{4}\right)\right]$$

$$= \frac{1}{2\pi}(2\pi^2) = \pi$$

and $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\cos x \sin nx)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin(1+n)x - \sin(1-n)x]\, dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right) - 1\cdot\left(-\frac{\sin(1+n)x}{(1+n)^2} + \frac{\sin(1\cancel{+}-n)x}{(1-n)^2}\right)\right]_0^{2\pi} (n \neq 1)$$

$$= \frac{1}{2\pi}\left[2\pi\left(-\frac{\cos 2(1+n)\pi}{1+n}\right\} + \frac{\cos 2(1-n)\pi}{1-n}\right)\Bigg]$$

$$= -\frac{1}{1+n} + \frac{1}{1-n} \quad [\because \cos 2(n+1)\pi = \cos 2(n-1)\pi = 1]$$

$$= \frac{2n}{1-n^2}, \quad (n \neq 1)$$

If $n = 1$, then

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\cos x \sin x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1 \cdot \left(\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(-\frac{2\pi}{2}+0\right) - (0+0)\right]$$

$$= -\frac{1}{2}$$

Substituting a_0, a_1, a_n, b_1 and b_n in ①, we get the required fourier series of $f(x)$ in the interval $(0, 2\pi)$ as

$$x\cos x = \pi\cos x - \frac{1}{2}\sin x + 2\sum_{n=2,3\ldots}^{\infty} \frac{n}{1-n^2} \sin nx$$

(or) $$x\cos x = \pi\cos x - \frac{1}{2}\sin x - 2\sum_{n=2,3\ldots}^{\infty} \frac{n}{n^2-1} \sin nx$$

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problems from Topic-IV

1. (i) Express $f(x) = x$ as a fourier series in $(-\pi, \pi)$.

Sol) Since $f(-x) = -x = -f(x)$

$\therefore f(x)$ is an odd function in $(-\pi, \pi)$

Hence in its fourier series expansion, the cosine terms are absent and sine terms are present.

$$\therefore x = \sum_{n=1}^{\infty} b_n \sin nx \text{ - - - - } ①$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$

$$= \frac{-2}{n}\cos n\pi = (-1)^{n+1}\frac{2}{n} \quad (\because \sin n\pi = 0)$$

Substituting the value of b_n in ①, we get.

$$x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right)$$

which is the required fourier series.

2. Expand the function $f(x) = x^3$ as a fourier series in $-\pi < x \le \pi$.

Sol) Since $f(-x) = (-x)^3 = -f(x)$,

$f(x)$ is an odd function

$\therefore$ The fourier series of $f(x)$ is of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{.....} ①$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx \, dx$

$$= \frac{2}{\pi}\left[x^3\left(\frac{-\cos nx}{n}\right) - 3x^2\left(\frac{-\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right]_0^{\pi},$$

Using Bernoulli's Rule.

$$= \frac{2}{\pi}\left[\frac{-\pi^3}{n}\cos n\pi + \frac{6\pi}{n^3}\cos n\pi\right]$$

$$= 2\cos n\pi\left[\frac{-\pi^2}{n} + \frac{6}{n^3}\right]$$

$$= 2(-1)^{n+1}\left[\frac{\pi^2}{n} - \frac{6}{n^3}\right]$$

Substituting (2) in (1), we get

$$f(x) = x^3 = 2\sum_{n=1}^{\infty}(-1)^{n+1}\left[\frac{\pi^2}{n} - \frac{6}{n^3}\right]\sin nx$$

3. Find the Fourier series for the function $f(x) = |x|$ in $-\pi < x < \pi$ and deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Sol) Since $f(-x) = |-x| = x = |x| = f(x)$, therefore $f(x) = |x|$ is an even function. Ⓣ

Here the fourier series will consist of ~~cos~~ cosine terms only.

$$\therefore f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \ldots ①$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$.

$$= \frac{2}{\pi}\int_0^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} dx.$$

$$= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \frac{2}{\pi}\left(\frac{\pi^2}{2}\right) = \pi \quad \ldots ②$$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} |x| \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\cdot\frac{\sin nx}{n} - 1\cdot\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}[(-1)^n - 1]$$

$$\therefore a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{-4}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases} \quad \ldots ③$$

Substituting the values of a_0 and a_n from (2) and (3) in (1), we get.

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) \quad \cdots ④$$

Deduction :

when $x = 0, |x| = |0| = 0$

∴ Putting $x = 0$ in ④, we have.

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \quad = \frac{\pi^2}{8}$$

Problems from Topic - VI :-

1) Obtain the Fourier cosine series for

$f(x) = \sin x \quad 0 < x < \pi$

Sol The half range Fourier cosine series is given by

$$f(x) = \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

here $a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x \, dx \qquad a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx$

$$a_0 = \frac{2}{\pi}(-\cos x)_0^{\pi}$$

$$= -\frac{2}{\pi}(\cos x)_0^{\pi} = -\frac{2}{\pi}(-1-1) = 4/\pi$$

$$\boxed{a_0 = 4/\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx = \frac{1}{\pi}\int_0^{\pi} 2\sin x \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} [\sin(1+n)x + \sin(1-n)x] \, dx$$

$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$= -\frac{1}{\pi}\left[\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left\{\underbrace{\left[\frac{\cos(1+n)\pi}{1+n} + \frac{\cos(1-n)\pi}{1-n}\right]}_{u.l} - \underbrace{\left[\frac{1}{1+n} + \frac{1}{1-n}\right]}_{l.l}\right\} \quad n \neq 1$$

$$= -\frac{1}{\pi}\left[\left(\frac{1}{1+n} + \frac{1}{1-n}\right)(\cos\pi \cos n\pi - 1)\right]$$

$$= \frac{2}{\pi(n^2-1)}[-\cos n\pi - 1] = \frac{-2}{\pi(n^2-1)}[(-1)^n + 1] \quad n \neq 1$$

$$a_n = 0 \qquad \text{n is odd}$$

$$= -4/(n^2-1)\pi \qquad \text{n is even}$$

if $n=1$ $\quad a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx$

$$= \frac{1}{\pi}\int_0^{\pi} 2\sin x \cos x \, dx$$

$$\Rightarrow \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = \frac{-1}{2\pi}\left[\cos 2\pi - \cos 0\right]$$

$$= \frac{-1}{2\pi}(+1-1) = 0$$

$a_1 = 0$

$$\therefore \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{a_0}{2} + \underset{n=1}{a_1 \cos 1x} + \underset{(n\neq 1)}{\sum_{n=2}^{\infty}} a_n \cos nx$$

$$= \frac{a_0}{2} + 0\cos x + \sum_{n=2}^{\infty}\left(\frac{-4}{\pi(n^2-1)}\right)\cos nx$$

$$\boxed{\sin x = \frac{4}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty}\left(\frac{\cos nx}{n^2-1}\right)}$$

② Expand $f(x) = \cos x$ $0<x<\pi$ in half range sine series

③ Find half range Fourier sine series of $f(x) = \pi - x$ in $[0,\pi]$

④ obtain the Fourier cosine series of $f(x) = e^{-x}$ in $0<x<\pi$

⑤ obtain Fourier cosine series for $f(x) = \pi - x$ $0<x<\pi$

Topic - IV Half range sine series & cosine series

Q.1) Find the half range sine series for $f(x) = x(\pi - x)$ in $0 < x < \pi$. Deduce that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots = \frac{\pi^3}{32}$

Sol: The fourier sine series expansion of $f(x)$ in $(0, \pi)$ is

$$f(x) = x(\pi - x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$

Hence $b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx\, dx$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4}{\pi n^3}\left[1 - (-1)^n\right]$$

$$\therefore b_n = \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{8}{\pi n^3}, & \text{when } n \text{ is odd} \end{cases}$$

Hence $x(\pi - x) = \sum_{n=1,3,5,..}^{\infty} \frac{8}{\pi n^3}\sin nx$

or $x(\pi - x) = \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \dots\right) \longrightarrow ①$

which is the required Fourier Sine series

Deduction:

Putting $x = \frac{\pi}{2}$ in ①, we get

$$\frac{\pi}{2}\left(x - \frac{\pi}{2}\right) = \frac{8}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3^3}\sin\frac{3\pi}{2} + \frac{1}{5^3}\sin\frac{5\pi}{2} + \dots\right)$$

$$\frac{\pi^2}{4} = \frac{8}{\pi}\left[1 + \frac{1}{3^3}\sin\left(\pi + \frac{\pi}{2}\right) + \frac{1}{5^3}\sin\left(2\pi + \frac{\pi}{2}\right) + \frac{1}{7^3}\sin\left(3\pi + \frac{\pi}{2}\right) + ..\right]$$

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots$$

Q.2) Find cosine and sine series for $f(x) = \pi - x$ in $[0, \pi]$

Sol: <u>Cosine Series</u>:

(a) The half-range cosine series of $f(x) = \pi - x$ in $[0, \pi]$ is

$$f(x) = \pi - x = \frac{a_0}{2} + \sum_{n-1}^{\infty} a_n \cos nx \longrightarrow ①$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\,dx$

$$= \frac{2}{\pi}\left(\pi x - \frac{x^2}{2}\right)_0^{\pi} = \frac{2}{\pi}\left[\left(\pi^2 - \frac{\pi^2}{2}\right) - (0-0)\right] = \frac{2}{\pi}\left(\frac{\pi^2}{2}\right) = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\frac{-\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[\left\{0 - \frac{\cos n\pi}{n^2}\right\} - \left\{0 - \frac{1}{n^2}\right\}\right]$$

$$= \frac{2}{\pi}\left(\frac{1 - \cos n\pi}{n^2}\right) = \frac{2}{\pi n^2}\left[1 - (-1)^n\right]$$

$$\therefore a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}$$

Substituting the values of a_0 and a_n in ①, we get

$$\pi - x = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1,3,5}^{\infty} \frac{1}{n^2}\cos nx$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right]$$

(b) <u>Sine</u> <u>Series</u>:

The half range sine series of $f(x) = \pi - x$ in $[0, \pi]$ is

$$f(x) = \pi - x = \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow ②$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} (\pi - x) \sin nx \, dx$

$$= \frac{2}{\pi}\left[(\pi - x)\frac{-\cos nx}{n} - (-1)\frac{-\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{-2}{\pi}\left[(\pi - x)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = \frac{-2}{\pi}\left[\{0+0\} - \{\frac{\pi}{n} + 0\}\right]$$

$$= \frac{2}{n}$$

Substituting value of b_n in ②, we get

$$\pi - x = 2\sum_{n=1}^{\infty} \frac{1}{n} \sin nx = 2\left[\sin nx + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right]$$

Q. Obtain the Fourier cosine series for $f(x) = x \sin x, 0 < x < \pi$
3) and show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots = \frac{\pi - 2}{4}$

<u>Sol</u>: Let $f(x) = x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \longrightarrow ①$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi} x \sin x \, dx$

$$= \frac{2}{\pi}\left[x(-\cos x) + \sin x\right]_0^{\pi} = \frac{2}{\pi}\left[-\pi\cos\pi + \sin\pi\right] = \frac{2}{\pi}(\pi) = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$= \frac{1}{\pi}\left\{x\left[\frac{-\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right]-(1)\left[\frac{-\sin(n+1)x}{(n+1)^2}+\frac{\sin(n-1)x}{(n-1)^2}\right]\right\}_0^{\pi}$$

$(n \neq 1)$

$$= \frac{-1}{n+1}\cos(n+1)\pi+\frac{1}{n-1}\cos(n-1)\pi$$

$$= \frac{(-1)^{n-1}}{n-1}-\frac{(-1)^{n+1}}{n+1} = \frac{2(-1)^{n+1}}{n^2-1} \quad (n \neq 1)$$

$$\therefore\ a_2 = -\frac{2}{1\cdot 3},\quad a_3 = \frac{2}{2\cdot 4},\quad a_4 = \frac{-2}{3\cdot 5},\quad a_5 = \frac{2}{4\cdot 6}$$

Now, $a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin x\,dx$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right)-\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{-\pi}{2}\cos 2\pi\right] = \frac{-1}{2}$$

From ①, we have

$$x\sin x = 1-\frac{1}{2}\cos x-\frac{2}{1\cdot 3}\cos 2x+\frac{2}{2\cdot 4}\cos 3x-\frac{2}{3\cdot 5}\cos 4x+\dots \rightarrow ②$$

Deduction:

Putting $x = \frac{\pi}{2}$ in ②, we obtain

$$\frac{\pi}{2} = 1+\frac{2}{1\cdot 3}-\frac{2}{3\cdot 5}+\frac{2}{5\cdot 7}-\frac{2}{7\cdot 9}+\dots$$

$$\frac{2}{1\cdot 3}-\frac{2}{3\cdot 5}+\frac{2}{5\cdot 7}-\frac{2}{7\cdot 9}+\dots = \frac{\pi}{2}-1 = \frac{\pi-2}{2}$$

$$\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\frac{1}{7\cdot 9}+\dots = \frac{\pi-2}{4}$$

The above result can also be written as

$$\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\frac{1}{7\cdot 9} = \frac{\pi}{4}-\frac{2}{4} = \frac{\pi}{4}-\frac{1}{2}$$

$$\Rightarrow \frac{1}{2}+\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\frac{1}{7\cdot 9}+\dots = \frac{\pi}{4}$$

(15)

Q.4) Find half range Fourier sine series of $f(x) = \pi - x$ in $[0, \pi]$

Sol: Refer Q.2 Part (b)

Q.5) Expand $f(x) = \cos x$, $0 < x < \pi$ in half range sine series

(or) Obtain the Fourier sine series for $f(x) = \cos x$

Sol: Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow$ ①

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\sin nx \cos x\,dx = \frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x + \sin(n-1)x]dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \quad (n \neq 1)$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^n}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right] \quad (n \neq 1)$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(-1)^2}{n+1} + \frac{(-1)^n}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left\{\frac{1}{n+1} + \frac{1}{n-1}\right\} + \left\{\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[\{(-1)^n + 1\}\left(\frac{1}{n+1} + \frac{1}{n-1}\right)\right] \quad (n \neq 1)$$

$$= \frac{2n}{\pi}\left[\frac{1+(-1)^n}{n^2-1}\right] \quad (n \neq 1)$$

$$\therefore b_n = \begin{cases} 0, & \text{when } n \text{ is odd}, n \neq 1 \\ \frac{4n}{\pi(n^2-1)}, & \text{when } n \text{ is even} \end{cases}$$

If $n=1$, then

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \cos x \sin x \, dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{1}{\pi}\left(\frac{-\cos 2x}{2}\right)_0^{\pi} = \frac{-1}{2\pi}(\cos 2\pi - \cos 0) = \frac{-1}{2\pi}(1-1) = 0$$

Thus $b_n = \begin{cases} 0 & \text{, when } n \text{ is odd} \\ \frac{4n}{\pi(n^2-1)} & \text{, when } n \text{ is even} \end{cases}$

Substituting values of b's in ①, we get

$$f(x) = \sum_{n=2,4,6\ldots}^{\infty} \frac{4n}{\pi(n^2-1)} \sin nx = \frac{4}{\pi}\sum_{n=2,4,6\ldots}^{\infty} \frac{n}{n^2-1} \sin nx$$

i.e, $\cos x = \frac{4}{\pi}\sum_{n=2,4,6\ldots}^{\infty} \frac{n}{n^2-1} \sin nx$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{2n}{4n^2-1} \sin 2nx \quad (\because n \text{ is even, replace } n \text{ by } 2n)$$

$$= \frac{8}{\pi}\left(\frac{1}{3}\sin 2x + \frac{2}{15}\sin 4x + \frac{3}{35}\sin 6x + \cdots\right)$$

TOPIC – 5 $\boxed{V}$ $\overset{I}{[0 \quad 2\pi]} \rightarrow [0 \quad \overset{IV \quad V}{2l}]$ ⑯

Q.1. Find the Fourier series with period 3 to represent $f(x) = x + x^2$ in $(0, 3)$

Sol: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \longrightarrow ①$

Here $2l = 3 \quad \therefore l = 3/2$

Hence the required Fourier series is

$$f(x) = x + x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2n\pi x}{3} + b_n \sin\frac{2n\pi x}{3}\right) \longrightarrow ②$$

where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{2}{3}\int_0^3 (x+x^2)\,dx = \frac{2}{3}\left(\frac{x^2}{2} + \frac{x^3}{3}\right) = 9$

$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{3}\int_0^3 (x+x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$

Integrating by parts, we obtain

$$a_n = \frac{2}{3}\left[\frac{3}{4n^2\pi^2} - \frac{9}{4n^2\pi^2}\right] = \frac{2}{3}\left(\frac{54}{4n^2\pi^2}\right) = \frac{9}{n^2\pi^2}$$

Finally, $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$

$$= \frac{2}{3}\int_0^3 (x+x^2)\sin\left(\frac{2n\pi x}{3}\right)dx = \frac{-12}{n\pi}$$

Substituting the values of a's and b's in ②, we get

$$x + x^2 = \frac{9}{2} + \frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) - \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{3}\right)$$

Q.2. Find the Fourier series of periodicity 3 for $f(x)=2x-x^2$ in $0<x<3$.

Sol: Here $2l = 3$, $l=\frac{3}{2}$

∴ The required Fourier series is of the form

$$f(x) = 2x-x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{3}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{3}\right) \longrightarrow ①$$

Then $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{2}{3}\int_0^{3}(2x-x^2)\,dx$

$$= \frac{2}{3}\left(x^2 - \frac{x^3}{3}\right)_0^3 = \frac{2}{3}(9-9) = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{3}\int_0^{3}(2x-x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(2x-x^2)\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)} - (2-2x)\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2} + (-2)\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right]_0^3$$

$$= \frac{2}{3}\left[\left\{0 - \frac{4\times 9}{4n^2\pi^2} + 0\right\} - \left\{0 + \frac{2\times 9}{4n^2\pi^2} + 0\right\}\right]$$

$$= \frac{-2}{3}\left[\frac{36+18}{4n^2\pi^2}\right] = \frac{-9}{n^2\pi^2}$$

Finally, $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{3}\int_0^{3}(2x-x^2)\sin\frac{2n\pi x}{3}\,dx$

$$b_n = \frac{2}{3}\left[(2x-x^2)\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)} - (2-2x)\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2} + (-2)\frac{\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{-3}{2n\pi}(-3)\right] = \frac{3}{n\pi}$$

. Substituting the values of a's and b's in ①, we get

$$2x - x^2 = -\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{3}\right)$$

$$= -\frac{9}{\pi^2}\left[\cos\frac{2\pi x}{3} + \frac{1}{2^2}\cos\frac{4\pi x}{3} - \frac{1}{3^2}\cos\frac{6\pi x}{3} + \dots\right)$$

$$+ \frac{3}{\pi}\left(\sin\frac{2\pi x}{3} + \frac{1}{2}\sin\frac{4\pi x}{3} + \frac{1}{3}\sin\frac{6\pi x}{3} + \dots\right)$$

TOPIC - VI [TOPIC - II → TOPIC - VI]

Q. Find the Fourier series expansion for f(x) ~~=~~, if

$$f(x) = \begin{cases} 2, & \text{if } -2 \leq x \leq 0 \\ x, & \text{if } 0 < x < 2 \end{cases}$$

Sol: Here $l = 2$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{b=1}^{\infty} b_n \sin\frac{n\pi x}{2}$ $(\because l = 2) \rightarrow$ ①

Then. $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} 2\,dx + \int_{0}^{2} x\,dx\right] = \frac{1}{2}\left[2(x)_{-2}^{0} + \left(\frac{x^2}{2}\right)_0^2\right] = 3$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx \quad (\because l = 2)$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[2\cdot\left(\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right)_{-2}^{0} + \left\{x\left(\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right) - 1\left(-\frac{\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}\right)\right\}_0^2\right]$$

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}\right]$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\therefore\ a_n = \begin{cases} 0 & , \text{when } n \text{ is even} \\ \frac{-4}{n^2\pi^2} & , \text{when } n \text{ is odd} \end{cases}$$

Finally,

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\left[\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[2\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)_{-2}^{0} + \left\{x\left(-\frac{\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right) - 1\left(-\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}\right)\right\}_{0}^{2}\right]$$

$$= \frac{1}{2}\left[\frac{-4}{n\pi} + \frac{4}{n\pi}\cos n\pi\right] + \frac{1}{2}\left[\frac{-4}{n\pi}\cos n\pi + \frac{4}{n^2\pi^2}\sin n\pi\right]$$

$$= \frac{1}{2}\left(\frac{-4}{n\pi}\right) = \frac{-2}{n\pi}$$

Substituting the value of a_0, a_n, b_n in ①, we get

$$f(x) = \frac{3}{2} - \frac{4}{\pi^2}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{2}\right)$$

Q. Find the Fourier series of the Function

$$f(x) = \begin{cases} \frac{1}{2} + x & , \text{when } -1 \leq x \leq 0 \\ \frac{1}{2} - x & , \text{when } 0 \leq x \leq 1 \end{cases}$$

Sol: Do it as a homework

· TOPIC – 7 $\overline{\text{VII}}$ TOPIC – $\overline{\text{III}}$ to TOPIC – $\overline{\text{VII}}$ ⑱ $[-l\ l]$

Q. Expand $f(x) = 3x^2 - 2$ as a Fourier series in the interval $(-3, 3)$

Sol: Since $f(x) = 3x^2 - 2$ is an even function

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Here $l = 3$

Hence the required series is of the form

$$f(x) = 3x^2 - 2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{3} \longrightarrow ①$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{3}\int_0^3 (3x^2-2)\,dx$$

$$= \frac{2}{3}\left(x^3 - 2x\right)_0^3 = \frac{2}{3}(27-6) = 14 \longrightarrow ②$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{3}\int_0^3 (3x^2-2)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{2}{3}\left[(3x^2-2)\frac{\sin\frac{n\pi x}{3}}{\frac{n\pi}{3}} - 6x\left(-\frac{\cos\frac{n\pi x}{3}}{\frac{n^2\pi^2}{9}}\right) + 6\left(-\frac{\sin\frac{n\pi x}{3}}{\frac{n^3\pi^3}{27}}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{18\times 9}{n^2\pi^2}\cos n\pi\right] = \frac{108(-1)^n}{n^2\pi^2} \longrightarrow ③$$

Substituting ② and ③ in ①

$$3x^2 - 2 = 7 + \sum_{n=1}^{\infty}\frac{108(-1)^n}{n^2\pi^2}\cos\frac{n\pi x}{3} = 7 - \frac{108}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{3}$$

$$= 7 - \frac{108}{\pi^2}\left[\cos\frac{\pi x}{3} - \frac{1}{4}\cos\frac{2\pi x}{3} + \frac{1}{9}\cos\pi x - \dots\right]$$

which is the required fourier series

Q. Find the Fourier series to represent $f(x) = x^2 - 2$, when $-2 \leq x \leq 2$

Sol: $f(x) = x^2 - 2$ is an even function. Here $l = 2$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} \rightarrow ①$

Then $a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{2}\int_0^2 (x^2-2)\,dx$

$$= \left(\frac{x^3}{3} - 2x\right)_0^2 = \frac{8}{3} - 4 = \frac{-4}{3}$$

$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \int_0^2 (x^2-2)\cos\frac{n\pi x}{2}\,dx \quad (\because l=2)$

$$= \left[(x^2-2)\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} - 2x\left(-\frac{\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}\right) + 2\left(-\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n^3\pi^3}{8}}\right)\right]_0^2$$

$$= \left\{\left[0 + \frac{16}{n^2\pi^2}\cos n\pi + 0\right] - [0+0-0]\right\}$$

$\therefore a_n = \frac{16}{n^2\pi^2}\cos n\pi = (-1)^n \frac{16}{n^2\pi^2}$

Substituting the values of a_0 and a_n in ①, we get

$$f(x) = -\frac{2}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}\cos\frac{n\pi x}{2}$$

i.e, $x^2 - 2 = \frac{-2}{3} - \frac{16}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}\cos\frac{n\pi x}{2}$

$$= \frac{-2}{3} - \frac{16}{\pi^2}\left(\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\pi x + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \frac{1}{4^2}\cos 2\pi x + \dots\right)$$

⑲

Q. If $f(x) = |x|$, expand $f(x)$ as a Fourier series in the interval $(-2, 2)$

Sol: Here $l = 2$, Since $|x|$ is an even function

$\therefore$ The required series is of the form $|x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} \rightarrow ①$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \int_0^2 |x|\,dx \quad (\because l = 2)$

$$= \int_0^2 x\,dx = \left.\frac{x^2}{2}\right]_0^2 = \frac{1}{2}(4-0) = 2 \longrightarrow ②$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{2}\,dx = \int_0^2 |x|\cos\frac{n\pi x}{2}\,dx \quad (\because l=2)$$

$$= \int_0^2 x\cos\frac{n\pi x}{2}\,dx \quad (\because 0<x<2)$$

$$= \left[x \cdot \frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)} - 1\left(-\frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n^2\pi^2}{4}\right)}\right)\right]_0^2$$

$$= \left(0 + \frac{\cos n\pi}{\left(\frac{n^2\pi^2}{4}\right)}\right) - \left(0 + \frac{1}{\left(\frac{n^2\pi^2}{4}\right)}\right)$$

$$= \frac{(-1)^n - 1}{\left(\frac{n^2\pi^2}{4}\right)} = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{-8}{n^2\pi^2}, & \text{when } n \text{ is odd} \end{cases} \rightarrow ③$$

Substituting ② and ③ in ①, we get

$$|x| = 1 - \frac{8}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2}\cos\frac{n\pi x}{2} = 1 - \frac{8}{\pi^2}\left[\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \ldots\right]$$

Half range sine & cosine series $(0, l)$

Q. Find the half-range cosine series for the function $f(x) = (x-1)^2$ in the interval $0 < x < 1$. Hence show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

Sol: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \quad (\because l = 1) \longrightarrow ①$

Then $a_0 = 2\int_0^1 f(x)\,dx \quad (\because l = 1)$

$$= 2\int_0^1 (x-1)^2 dx = 2\left[\frac{(x-1)^3}{3}\right]_0^1 = \frac{2}{3}(0+1) = \frac{2}{3}$$

$a_n = 2\int_0^1 f(x) \cos n\pi x\,dx \quad (\because l = 1)$

$= 2\int_0^1 (x-1)^2 \cos n\pi x\,dx$. Apply Bernoulli's Rule

$$= 2\left[(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left\{(0+0-0) - \left(0 - \frac{2}{n^2\pi^2} - 0\right)\right\} = \frac{4}{n^2\pi^2}$$

Substituting values of a_0 and a_n in ①, we get

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi x$$

i.e, $(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\left(\frac{1}{1^2}\cos \pi x + \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x + \dots\right) \to ②$

Deduction: Put $x = 0$ in ②, we obtain

$$1 - \frac{1}{3} = \frac{4}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}.$$

⑳

Section - B

Fouriers transform

Topic-I: Fourier Integrals ④

Topic-II: fourier transform and Inverse Fourier transform ④

Topic-III: Fourier sine transform. and Fourier cosine transform — ⑨

Topic-IV: Finite Fourier transform — ⑥

Imp formulae:

① Fourier sine Integral of f(x)

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\underset{(\lambda)}{\lambda x}\int_0^\infty f(t)\sin\lambda t\,dt\,d\lambda$$

② Fourier cosine Integral of f(x)

$$f(x)=\frac{2}{\pi}\int_0^\infty \cos\underset{(\lambda)}{\lambda x}\int_0^\infty f(t)\cos\lambda t\,dt\,d\lambda$$

③ Fourier Integral in complex form

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ip(t-x)} f(t)\,dt\,dp$$

④ Fourier transform of f(x)

$$F(f(x) = \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$$

⑤ Inverse Fourier transform

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx} F(f(x))\,dp$$

⑤ Fourier cosine transform

$$F_c(f(x))= \int_0^\infty f(x)\cos px\,dx$$

⑥ Inverse Fourier cosine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(f(x))\cos px\,dp$$

[c stands for cosine]

7) Fourier ~~co~~sine transform

$$F_s(f(x)) = \int_0^\infty f(x)\,\overset{\sin}{\cancel{\cos}}\, px\,dx$$

8) Inverse Fourier ~~co~~sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(f(x))\sin px\,dp$$

[S – stands for sine.]

⑨ Finite Fourier sine transform

$$F_s(f(x)) = F_s(n) = \int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

⑩ Finite Fourier cosine transform

$$F_c(f(x)) = F_c(n) = \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

⑪ $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right]$$

⑫ $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx - b\cdot\sin bx\right)$$

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Topic-I Fourier Integrals

Fourier Sine Integrals & Fourier cosine Integrals

1. Using Fourier integral show that

$$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^{\infty} \frac{\lambda \sin\lambda x \, d\lambda}{(\lambda^2+a^2)(\lambda^2+b^2)}, \quad a,b>0$$

W.K.T Fourier sine integral for $f(x)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \sin px \int_0^{\infty} f(t)\sin pt \, dt \, dp$$

Replacing p with λ, we get

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \sin\lambda x \int_0^{\infty} f(t)\sin\lambda t \, dt \, d\lambda \rightarrow ①$$

Here $f(x) = e^{-ax} - e^{-bx}$; $f(t) = e^{-at} - e^{-bt} \rightarrow ②$

Sub eq ② in eq ①, we get

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \sin\lambda x \left\{\int_0^{\infty} (e^{-at} - e^{-bt}) \sin\lambda t \, dt\right\} d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \sin\lambda x \left\{\int_0^{\infty} (e^{-at}\sin\lambda t) dt - \int_0^{\infty} e^{-bt}\sin\lambda t \, dt\right\} d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \sin\lambda x \left[\left\{\frac{e^{-at}}{\lambda^2+a^2}(-a\sin\lambda t - \lambda\cos\lambda t)\right\}_0^{\infty} - \left\{\frac{e^{-bt}}{\lambda^2+b^2}(-b\sin\lambda t - \lambda\cos\lambda t)\right\}_0^{\infty}\right]$$

Using $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{\lambda}{\lambda^2+a^2} - \frac{\lambda}{\lambda^2+b^2}\right]d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\cdot\lambda\left[\frac{1}{\lambda^2+a^2} - \frac{1}{\lambda^2+b^2}\right]d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\cdot\frac{\lambda(b^2-a^2)}{(\lambda^2+a^2)(\lambda^2+b^2)}\,d\lambda$$

$$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}\,d\lambda$$

2. Using Fourier integral, S.T $\int_0^\infty \frac{1-\cos\pi\lambda}{\lambda}\sin x\lambda\,d\lambda = \begin{cases}\frac{\pi}{2}, & \text{if } 0<x<\pi\\ 0, & \text{if } x>\pi\end{cases}$

Sol Let $f(x) = \begin{cases}\frac{\pi}{2}, & \text{if } 0<x<\pi\\ 0, & \text{if } x>\pi\end{cases}$; then $f(t) = \begin{cases}\frac{\pi}{2}, & \text{if } 0<t<\pi\\ 0, & \text{if } t>\pi\end{cases} \rightarrow ①$

W.K.T fourier sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\int_0^\infty f(t)\sin\lambda t\,dt\,d\lambda \rightarrow ②$$

Sub eq ① in eq ②, we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\int_0^\pi f(t)\sin\lambda t\,dt\right]d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\int_0^\pi \frac{\pi}{2}\cdot\sin\lambda t\,dt\right]d\lambda$$

$$= \frac{2}{\pi}\cdot\frac{\pi}{2}\int_0^\infty \sin\lambda x\left(-\frac{\cos\lambda t}{\lambda}\right)_0^\pi d\lambda$$

$$= \int_0^\infty \sin\lambda x\left(\frac{1-\cos\lambda\pi}{\lambda}\right)d\lambda$$

$$\therefore \int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}\sin x\lambda\,d\lambda = \begin{cases}\frac{\pi}{2}, & \text{if } 0<x<\pi\\ 0, & \text{if } x>\pi\end{cases}$$

3. Using Fourier Integral, S.T $e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\lambda x}{\lambda^2+a^2}\,d\lambda$ $(a>0, x\geq 0)$

Sol: By Fourier cosine integral formula.

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos px \int_0^\infty f(t)\cos pt\,dt\,dp$$

Replacing 'p' with 'λ', we get

$$f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\cos\lambda t\cos\lambda x\,dt\,d\lambda$$

Let $f(x) = e^{-ax}$. Then,

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty\int_0^\infty e^{-at}\cos\lambda t\cos\lambda x\,dt\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\int_0^\infty e^{-at}\cos\lambda t\,dt\right]d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x \cdot \frac{a}{a^2+\lambda^2}\,d\lambda = \frac{2a}{\pi}\int_0^\infty \frac{\cos\lambda x}{\lambda^2+a^2}\,d\lambda$$

4. Using Fourier Integral, S.T $e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{\lambda^2+2}{\lambda^4+4}\cos\lambda x\,d\lambda$

Sol $$f(x) = \frac{2}{\pi}\int_0^\infty \cos px\int_0^\infty f(t)\cos pt\,dt\,dp$$

Replacing 'p' with 'λ' we get

$$f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\cos\lambda t\cos\lambda x\,dt\,d\lambda$$

Let $f(x) = e^{-x}\cos x$. Then,

$$e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty\int_0^\infty e^{-t}\cos t \;\cos\lambda t\;\cos\lambda x\; dt\; d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty\left[\int_0^\infty e^{-t}(2\cos t\cos\lambda t)\,dt\right]\cos\lambda x\; d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty\left[\int_0^\infty e^{-t}\left\{\cos(\lambda+1)t+\cos(\lambda-1)t\right\}dt\right]\cos\lambda x\; d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty\left[\frac{1}{(\lambda+1)^2+1}+\frac{1}{(\lambda-1)^2+1}\right]\cos\lambda x\; d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty\left[\frac{1}{\lambda^2+2\lambda+2}+\frac{1}{\lambda^2-2\lambda+2}\right]\cos\lambda x\; d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty\left[\frac{2(\lambda^2+2)\cos\lambda x}{[(\lambda^2+2)+2\lambda][(\lambda^2+2)-2\lambda]}\right]\cos\lambda x\; d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty\frac{(\lambda^2+2)\cos\lambda x}{(\lambda^2+2)^2-(2\lambda)^2}\; d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty\frac{\lambda^2+2}{\lambda^4+4}\;\cos\lambda x\; d\lambda$$

Topic-II Fourier transform & Inverse Fourier transform

1. Find the Fourier transform of $f(x)$ defined by $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$

and hence evaluate $\int_0^\infty \frac{\sin p}{p} dp$ or $\int_0^\infty \frac{\sin x}{x} dx$ and

$\int_{-\infty}^{\infty} \frac{\sin ap \cdot \cos px}{p} dp$

Sol:
$$F[f(x)] = \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$$

$$= \int_{-\infty}^{-a} e^{ipx} f(x)\, dx + \int_{-a}^{a} e^{ipx} f(x)\, dx + \int_{a}^{\infty} e^{ipx} f(x)\, dx$$

$$= \int_{-a}^{a} e^{ipx} (1)\, dx = \left[\frac{e^{ipx}}{ip}\right]_{-a}^{a} = \frac{e^{ipa} - e^{-ipa}}{ip} = \frac{2}{ip} \sinh(ipa)$$

$$\therefore F[f(x)] = F(P) = \frac{2\sin pa}{p} \qquad [\because \sinh(ix) = i\sin x]$$

Second part:

w.k.T $F(P) = \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$, by the inversion formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p) e^{-ipx} dp$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin pa}{p} \cdot e^{-ipx} dp = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

$$L.H.S = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin pa}{p} \cos px\, dp - \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin pa}{p} \sin px\, dp$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin pa \cos px}{p} \cdot dp$$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin ap \cdot \cos px}{p}\, dp = \begin{cases} \pi, & |x| < a \\ 0, & |x| > a \end{cases} \rightarrow ①$$

Put $x = 0$ in eq ①, then $\displaystyle\int_{-\infty}^{\infty} \frac{\sin ap}{p}\, dp = \begin{cases} \pi, & \text{if } a > 0 \\ 0, & \text{if } a < 0 \end{cases}$

$$\text{or } 2\int_{0}^{\infty} \frac{\sin ap}{p}\, dp = \begin{cases} \pi, & \text{if } a > 0 \\ 0, & \text{if } a < 0 \end{cases}$$

$$\therefore \int_{0}^{\infty} \frac{\sin ax}{x}\, dx = \begin{cases} \frac{\pi}{2}, & \text{if } a > 0 \\ 0, & \text{if } a < 0 \end{cases}$$

Third part: If $x = 0$ and $a = 1$, then $\displaystyle\int_{-\infty}^{\infty} \frac{\sin p}{p}\, dp = \pi$

$$\text{or } \int_{0}^{\infty} \frac{\sin p}{p}\, dp = \frac{\pi}{2} \text{ or } \int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

2. Find the Fourier transform of $f(x)$ defined by

$$f(x) = \begin{cases} 1 - x^2, & \text{if } |x| \leq 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$

Hence evaluate

i) $\displaystyle\int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx$ ii) $\displaystyle\int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3}\, dx$ (or)

S.T $\displaystyle\int_{0}^{\infty} \frac{\sin x - x\cos x}{x^3}\, dx = \frac{\pi}{4}$

Sol: $\displaystyle F[f(x)] = \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$

$$= \int_{-\infty}^{-1} e^{ipx}.\, 0\, dx + \int_{-1}^{1} e^{ipx}(1 - x^2)\, dx + \int_{1}^{\infty} e^{ipx}.\, 0\, dx$$

$$= \int_{-1}^{1} (1 - x^2) e^{ipx}\, dx = \left[\frac{(1 - x^2) e^{ipx}}{ip}\right]_{-1}^{1} + 2\int_{-1}^{1} \frac{x e^{ipx}}{ip}\, dx$$

$$= \frac{2}{ip}\int_{-1}^{1} x e^{ipx}\, dx$$

$$= \frac{2}{ip}\left[\left(\frac{xe^{ipx}}{ip}\right)_{-1}^{1} - \int_{-1}^{1} 1\cdot\frac{e^{ipx}}{ip}\, dx\right]$$

$$= \frac{2}{ip}\left[\left(\frac{e^{ip}+e^{-ip}}{ip}\right) - \frac{1}{(ip)^2}\left(e^{ipx}\right)_{-1}^{1}\right] = \frac{2}{ip}\left[\frac{2\cos p}{ip} + \frac{1}{p^2}\left(e^{ip}-e^{-ip}\right)\right]$$

$$= \frac{2}{ip}\left[\frac{-2\cos p}{p^2} + \frac{2i\sin p}{ip^3}\right] = 4\left[\frac{-p\cos p+\sin p}{p^3}\right] = \frac{-4}{p^3}(p\cos p-\sin p)$$

Second part:

i) By inversion formula, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)\, e^{-ipx}\, dp \text{ or } \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-4}{p^3}(p\cos p-\sin p)\, e^{-ipx}\, dp = \begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases}$$

Put $x=\frac{1}{2}$, $\Rightarrow \frac{-1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{p^3}(p\cos p-\sin p)\, e^{-ip/2}\, dp = 1-\frac{1}{4} = \frac{3}{4}$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{1}{p^3}(p\cos p-\sin p)\left(\cos\frac{p}{2} - i\sin\frac{p}{2}\right) dp = \frac{-3\pi}{8}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{p\cos p-\sin p}{p^3}\cos\frac{p}{2}\, dp = \frac{-3\pi}{8}$$

$$\Rightarrow 2\int_{0}^{\infty}\frac{p\cos p-\sin p}{p^3}\cos\frac{p}{2}\, dp = \frac{-3\pi}{8} \quad \text{(even)}$$

$$\Rightarrow \int_{0}^{\infty}\frac{x\cos x-\sin x}{x^3}\cos\frac{x}{2}\, dx = \frac{-3\pi}{16}$$

Put $x=0 \Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-4}{p^3}(p\cos p-\sin p)\, dp = 1$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{p\cos p - \sin p}{p^3}\, dp = \frac{-\pi}{2} \quad \Rightarrow 2\int_{0}^{\infty} \frac{p\cos p - \sin p}{p^3}\, dp = \frac{-\pi}{2}$$

$$\therefore \int_{0}^{\infty} \frac{\sin x - x\cos x}{x^3}\, dx = \frac{\pi}{4}$$

3. Find the Fourier transform of $f(x)$ defined by.

$$f(x) = \begin{cases} e^{iqx}, & \alpha < x < \beta \\ 0, & x < \alpha \text{ and } x > \beta \end{cases} \quad \text{or} \quad f(x) = \begin{cases} e^{ikx}, & a < x < b \\ 0, & x < a \text{ and } x > b \end{cases}$$

Sol: $F[f(x)] = \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$

$$= \int_{\alpha}^{\beta} e^{ipx} e^{iqx}\, dx = \int_{\alpha}^{\beta} e^{i(p+q)x}\, dx = \left[\frac{e^{i(p+q)x}}{i(p+q)}\right]_{\alpha}^{\beta}$$

$$= \frac{e^{i(p+q)\beta} - e^{i(p+q)\alpha}}{i(p+q)}$$

4. Find Fourier transform of $f(x) = e^{-\frac{x^2}{2}}, -\infty < x < \infty$

(or) show that fourier transform of $e^{-\frac{x^2}{2}}$ is reciprocal

Sol: $F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{ipx} dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cdot e^{ipx} dx = \int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-ip)^2} e^{-\frac{p^2}{2}} dx = e^{\frac{-p^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-ip)^2} dx$

Put $\frac{1}{\sqrt{2}}(x-ip) = t \Rightarrow dx = \sqrt{2}\, dt$

$$F[f(x)] = e^{\frac{-p^2}{2}} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\, dt = \sqrt{2}\, e^{\frac{-p^2}{2}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \sqrt{2}\, e^{\frac{-p^2}{2}} \sqrt{\pi} = \sqrt{2\pi}\, e^{\frac{-p^2}{2}}$$

$\therefore$ Fourier transform of $e^{\frac{-x^2}{2}}$ is $e^{\frac{-p^2}{2}}$. Hence $f(x)$ is self-reciprocal

Topic-III Fourier sine & cosine transform and Inverse Fourier sine & cosine transform

1. Find the Fourier cosine transform of the function $f(x)$ defined by $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

Sol :

$$F_c[f(x)] = \int_0^\infty f(x)\cos px\,dx$$

$$= \int_0^a \cos x\,\cos px\,dx = \frac{1}{2}\int_0^a [\cos(1+p)x + \cos(1-p)x]\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(1+p)x}{1+p} + \frac{\sin(1-p)x}{1-p}\right]_0^a = \frac{1}{2}\left[\frac{\sin(1+p)a}{1+p} + \frac{\sin(1-p)a}{1-p}\right]$$

2. Find the Fourier sine and cosine transforms of $f(x) = \frac{e^{-ax}}{x}$ and deduce that $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\sin sx\,dx = \text{Tan}^{-1}\left(\frac{s}{a}\right) - \text{Tan}^{-1}\left(\frac{s}{b}\right)$

Sol: Fourier Sine Transform:

$$F_s[f(x)] = \int_0^\infty f(x)\sin px\,dx = \int_0^\infty \frac{e^{-ax}}{x}\sin px\,dx$$

Differentiating both sides w.r.t 'p', we get

$$\frac{d}{dp}\{F_s[f(x)]\} = \int_0^\infty \frac{x e^{-ax}\cos px}{x} = \int_0^\infty e^{-ax}\cos px\,dx = \frac{a}{p^2+a^2}$$

$$\therefore \frac{d}{dp}\{F_s[f(x)]\} = \frac{a}{p^2+a^2}$$

Integrating w.r.t 'p', we get

$$F_s[f(x)] = \int \frac{a}{p^2+a^2}\,dp = \text{Tan}^{-1}\left(\frac{p}{a}\right) + c$$

If $p = 0$, then $F_s[f(x)] = 0$ and $\therefore c = 0$

$\therefore F_S[f(x)] = \text{Tan}^{-1}\left(\frac{p}{a}\right) \Rightarrow F_S\left\{\frac{e^{-ax}}{x}\right\} = \text{Tan}^{-1}\left(\frac{p}{a}\right) \to Ⓐ$

W.K.T Fourier Sine transform of $f(x)$ is given by

$$F_S\{f(x)\} = \int_0^\infty f(x)\sin px\,dx = \int_0^\infty f(x)\sin sx\,dx \to ①$$

Let $f(x) = \dfrac{e^{-ax} - e^{-bx}}{x} \to ②$

substituting eq ② in eq ①. we get

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\sin px\,dx = F_S\left[\frac{e^{-ax} - e^{-bx}}{x}\right]$$

$$= F_S\left\{\frac{e^{-ax}}{x}\right\} - F_S\left\{\frac{e^{-bx}}{x}\right\}$$

$$= \text{Tan}^{-1}\left(\frac{p}{a}\right) - \text{Tan}^{-1}\left(\frac{p}{b}\right), \text{ from A}$$

$$\therefore \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\sin sx\,dx = \text{Tan}^{-1}\left(\frac{s}{a}\right) - \text{Tan}^{-1}\left(\frac{s}{b}\right)$$

Fourier cosine Transform:

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx = \int_0^\infty \frac{e^{-ax}}{x}\cos px\,dx$$

Differentiating on both sides w.r.t 'p', we get

$$\frac{d}{dp}\left[F_c\{f(x)\}\right] = \frac{d}{dp}\int_0^\infty \frac{e^{-ax}}{x}\cos px\,dx = \int_0^\infty \frac{\partial}{\partial p}\left\{\frac{e^{-ax}}{x}\cos px\right\}dx$$

$$= \int_0^\infty \frac{e^{-ax}}{x}(-\sin px)\,x\,dx = -\int_0^\infty e^{-ax}\sin px\,dx$$

$$= \frac{-p}{a^2+p^2} \qquad \left[\because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}\right]$$

Integrating w.r.t 'p', we get

$$F_c\{f(x)\} = -\int \frac{p}{a^2+p^2}\,dp = \frac{-1}{2}\int \frac{2p}{p^2+a^2}\,dp = \frac{-1}{2}\log(p^2+a^2)$$

$$\therefore\ F_c\left\{\frac{e^{-ax}}{x}\right\} = \frac{-1}{2}\log(p^2+a^2) = \log\left(\frac{1}{\sqrt{p^2+a^2}}\right)$$

Let $f(x) = \dfrac{e^{-ax} - e^{-bx}}{x}$. Then,

$$F_c\{f(x)\} = F_c\left\{\frac{e^{-ax}-e^{-bx}}{x}\right\} = F_c\left\{\frac{e^{-ax}}{x}\right\} - F_c\left\{\frac{e^{-bx}}{x}\right\}$$

$$= \frac{-1}{2}\log(p^2+a^2) + \frac{1}{2}\log(p^2+b^2)$$

$$= \frac{1}{2}\log\left(\frac{p^2+b^2}{p^2+a^2}\right)$$

3. Find Fourier cosine and sine transforms of e^{-ax}, $a>0$ and hence deduce the inversion formula

(OR) Deduce the integrals (i) $\int_0^\infty \frac{\cos px}{a^2+p^2}\,dp$ (ii) $\int_0^\infty \frac{p\sin px}{a^2+p^2}\,dp$

Sol: Let $f(x) = e^{-ax}$. Then,

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx = \int_0^\infty e^{-ax}\cos px\,dx$$

$$= \left\{\frac{e^{-ax}}{a^2+p^2}\left[-a\cos px + p\sin px\right]\right\}_0^\infty = \frac{a}{a^2+p^2}$$

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx = \int_0^\infty e^{-ax}\sin px\,dx$$

$$= \left\{\frac{e^{-ax}}{a^2+p^2}\left[-a\sin px - p\cos px\right]\right\}_0^\infty = \frac{p}{a^2+p^2}$$

i) By inverse Fourier cosine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c\{f(x)\}\cos px\,dp \;=\; \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+p^2}\cos px\,dp$$

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos px}{a^2+p^2}\,dp$$

$$\frac{\pi}{2a}e^{-ax} = \int_0^\infty \frac{\cos px}{a^2+p^2}\,dp$$

ii) By inverse Fourier sine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s\{f(x)\}\sin px\,dp = \frac{2}{\pi}\int_0^\infty \frac{p}{a^2+p^2}\sin px\,dp$$

$$\int_0^\infty \frac{p\sin px}{a^2+p^2}\,dp = \frac{\pi}{2}e^{-ax}$$

4. Find the Fourier Sine transform of $\frac{x}{a^2+x^2}$ and Fourier cosine transform of $\frac{1}{a^2+x^2}$

sol: we have $F_s\{e^{-ax}\} = \frac{p}{a^2+p^2}$

The inverse Fourier sine transform of e^{-ax} is

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty F_s\{e^{-ax}\}\sin px\,dp = \frac{2}{\pi}\int_0^\infty \frac{p}{a^2+p^2}\sin px\,dp$$

$$\Rightarrow \int_0^\infty \frac{p}{a^2+p^2}\sin px\,dp = \frac{\pi}{2}e^{-ax}$$

changing p to x and x to p, then $\int_0^\infty \frac{x}{a^2+x^2}\sin px\,dx = \frac{\pi}{2}e^{-ap}$

$$\therefore F_s\left\{\frac{x}{a^2+x^2}\right\} = \frac{\pi}{2}e^{-ap}$$

Also we have $F_c\{e^{-ax}\} = \frac{a}{a^2+p^2}$

Inverse Fourier cosine transform of e^{-ax} is

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty F_c\{e^{-ax}\}\cos px\,dp = \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+p^2}\cos px\,dp = \frac{2a}{\pi}\int_0^\infty \frac{\cos px}{a^2+p^2}\,dp$$

$$\therefore \int_0^\infty \frac{\cos px}{a^2+p^2}\,dp = \frac{\pi}{2a}e^{-ax}$$

changing p to x and x to p, then $\int_0^\infty \frac{1}{a^2+x^2}\cos px\,dx = \frac{\pi}{2a}e^{-ap}$

$$\boxed{F_c\left\{\frac{1}{a^2+x^2}\right\} = \frac{\pi}{2}e^{-ap}}$$

5. Find the Fourier sine and cosine transforms of $2e^{-5x}+5e^{-2x}$.

Sol: Let $f(x) = 2e^{-5x}+5e^{-2x}$

i) Fourier sine Transform of $f(x)$ is given by

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx = \int_0^\infty (2e^{-5x}+5e^{-2x})\sin px\,dx$$

$$= 2\int_0^\infty e^{-5x}\sin px\,dx + 5\int_0^\infty e^{-2x}\sin px\,dx$$

$$= 2\left[\frac{e^{-5x}}{p^2+25}(-5\sin px - p\cos px)\right]_0^\infty + 5\left[\frac{e^{-2x}}{p^2+4}(-2\sin px - p\cos px)\right]_0^\infty$$

$$= 2\left[0+\frac{1}{p^2+25}(0+P)\right]+5\left[0+\frac{1}{p^2+4}(0+P)\right] = \frac{2P}{p^2+25}+\frac{5P}{p^2+4}$$

ii) Fourier cosine Transform of $f(x)$ is given by

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx = \int_0^\infty (2e^{-5x}+5e^{-2x})\cos px\,dx$$

$$= 2\int_0^\infty e^{-5x}\cos px\,dx + 5\int_0^\infty e^{-2x}\cos px\,dx$$

$$= 2\left[\frac{e^{-5x}}{p^2+25}(-5\cos px + p\sin px)\right]_0^\infty + 5\left[\frac{e^{-2x}}{p^2+4}(-2\cos px + p\sin px)\right]_0^\infty$$

$$= \frac{10}{p^2+25} + \frac{10}{p^2+4}$$

6. Show that the Fourier sine transform of $f(x)=\begin{cases} x & \text{for } 0<x<1 \\ 2-x & \text{for } 1<x<2 \\ 0 & \text{for } x>2 \end{cases}$

is $\frac{2\sin s\,(1-\cos s)}{s^2}$

Sol: $F_s\{f(x)\} = \int_0^\infty f(x)\sin sx\,dx$

$$= \int_0^1 f(x)\sin sx\,dx + \int_1^2 f(x)\sin sx\,dx + \int_2^\infty f(x)\sin sx\,dx$$

$$= \int_0^1 x\cdot\sin sx\,dx + \int_1^2 (2-x)\sin sx\,dx$$

$$= \left(-x\,\frac{\cos sx}{s}\right)_0^1 + \int_0^1 \frac{\cos sx}{s}\,dx + \left[-(2-x)\frac{\cos sx}{s}\right]_1^2 + \int_1^2 (-1)\frac{\cos sx}{s}$$

$$= \frac{-\cos s}{s} + \frac{1}{s}\left(\frac{\sin sx}{s}\right)_0^1 + \left[0+\frac{\cos s}{s}\right] - \frac{1}{s}\left(\frac{\sin sx}{s}\right)_1^2$$

$$= \frac{1}{s^2}\sin s - \frac{1}{s^2}(\sin 2s - \sin s) = \frac{2\sin s - \sin 2s}{s^2}$$

$$= \frac{2\sin s(1-\cos s)}{s^2}$$

7. Find Fourier cosine transform f $f(x)$ is defined by

$$f(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2. \end{cases}$$

Sol: $F_c[f(x)] = \int_0^\infty f(x)\cos px\,dx = \int_0^1 x\cos px\,dx + \int_1^2 (2-x)\cos px\,dx + \int_2^\infty 0\cdot\cos px\,dx$

$$= \int_0^1 x\cos px\,dx + \int_1^2 (2-x)\cos px\,dx$$

$$= \left[x\left(\frac{\sin px}{p}\right) - 1\cdot\left(\frac{-\cos px}{p^2}\right)\right]_0^1 + \left[(2-x)\frac{\sin px}{p} - (-1)\left(\frac{-\cos px}{p^2}\right)\right]_1^2$$

$$= \left(\frac{\sin p}{p} + \frac{\cos p}{p^2} - 0 - \frac{1}{p^2}\right) + \left(0 - \frac{\cos 2p}{p^2} - \frac{\sin p}{p} + \frac{\cos p}{p^2}\right)$$

$$= \frac{2\cos p - \cos 2p - 1}{p^2} = \frac{2\cos p - (2\cos^2 p - 1) - 1}{p^2} = \frac{1}{p^2}(2\cos p - 2\cos^2 p)$$

$$= \frac{2}{p^2}\cos p(1-\cos p)$$

9. Find the Inverse fourier Cosine Transforms f(x) of

$$F_c(P) = \begin{cases} \frac{1}{2a}\left(a-\frac{P}{2}\right), & \text{when } p < 2a \\ 0 \quad , & \text{when } p \geq 2a \end{cases}$$

sol: From the Inverse Fourier cosine Transform, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(P)\cos px\,dp$$

$$= \frac{2}{\pi}\left[\frac{1}{2a}\int_0^{2a}\left(a-\frac{P}{2}\right)\cos px\,dp + \int_{2a}^{\infty} 0\cdot\cos px\,dp\right]$$

$$= \frac{2}{\pi}\cdot\frac{1}{2a}\int_0^{2a}\left(a-\frac{P}{2}\right)\cos px\,dp$$

$$= \frac{1}{a\pi}\int_0^{2a}\left(a-\frac{P}{2}\right)d\left(\frac{\sin px}{x}\right)$$

$$= \left[\frac{1}{a\pi}\left(a-\frac{P}{2}\right)\frac{\sin px}{x}\right]_0^{2a} - \frac{1}{a\pi}\int_0^{2a}\frac{\sin px}{x}\cdot\left(\frac{-1}{2}\right)dp$$

$$= 0 + \frac{1}{2a\pi}\int_0^{2a}\frac{\sin px}{x}\,dp$$

$$= \left[\frac{1}{2a\pi}\left(\frac{-\cos px}{x^2}\right)\right]_0^{2a}$$

$$= \left[\frac{-\cos px}{2a\pi x^2}\right]_0^{2a} = \frac{-1}{2a\pi x^2}\left[\cos 2ax - 1\right]$$

$$= \frac{1-\cos 2ax}{2a\pi x^2}$$

$$= \frac{\sin^2 ax}{a\pi x^2}$$

8. Find the Fourier cosine Transforms of

a) $e^{-ax}\cos ax$ b) $e^{-ax}\sin ax$

Sol: a) Let $f(x) = e^{-ax}\cos ax$

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx = \int_0^\infty e^{-ax}\cos ax\,\cos px\,dx$$

$$= \frac{1}{2}\int_0^\infty e^{-ax}\left[\cos(a+p)x + \cos(a-p)x\right]dx$$

$$= \frac{1}{2}\left[\frac{a}{a^2+(p+a)^2} + \frac{a}{a^2+(p-a)^2}\right]$$

$$= \frac{a}{2}\left[\frac{a^2+(p-a)^2+a^2+(p+a)^2}{[a^2+(p+a)^2]\,[a^2+(p-a)^2]}\right]$$

$$= \frac{a\,(p^2+2a^2)}{(p^2+2ap+2a^2)(p^2-2ap+2a^2)}$$

b) Let $g(x) = e^{-ax}\sin ax$

$$F_s\{g(x)\} = \int_0^\infty e^{-ax}\sin ax\,\sin px\,dx$$

$$= \frac{1}{2}\int_0^\infty e^{-ax}\left[\cos(a-p)x - \cos(a+p)x\right]dx$$

$$= \frac{1}{2}\left[\frac{a}{a^2+(p-a)^2} - \frac{a}{a^2+(p+a)^2}\right]$$

$$= \frac{a}{2}\left[\frac{a^2+(p+a)^2-a^2-(p-a)^2}{[a^2+(p-a)^2]\,[a^2+(p+a)^2]}\right]$$

$$= \frac{2a^2p}{(p^2-2ap+2a^2)(p^2+2ap+2a^2)}$$

Topic-IV Finite Fourier transforms

1. Find the finite Fourier Sine and cosine Transforms of $f(x)=1$

Sol: Here, the range of x is not given. We take the usual range $[0,\pi]$.

i) $$F_s\{f(x)\} = F_s(n) = \int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^\pi 1\cdot\sin\left(\frac{n\pi x}{\pi}\right)dx = \int_0^\pi \sin nx\,dx$$

$$= \left(-\frac{\cos nx}{n}\right)_0^\pi = \frac{-1}{n}(\cos n\pi - 1) = \frac{1-\cos n\pi}{n}$$

$$= \frac{1-(-1)^n}{n}$$

ii) $$F_c\{f(x)\} = F_c(n) = \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \int_0^\pi 1\cdot\cos\left(\frac{n\pi x}{n}\right)dx$$

$$= \left(\frac{\sin nx}{n}\right)_0^\pi = \frac{1}{n}(\sin n\pi - \sin 0)$$

If $n=0$ then $F_c\{f(x)\} = \int_0^\pi 1\cdot 1\,dx = \pi$

2. Find the finite Fourier sine and cosine Transforms of $f(x)$, defined by $f(x)=2x$, where $0<x<2\pi$

Sol: $$F_s(n) = \int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \int_0^{2\pi} 2x\cdot\sin\left(\frac{nx}{2}\right)dx$$

$$= 2\left[x\left(\frac{-\cos\frac{nx}{2}}{\frac{n}{2}}\right) - \left(\frac{-\sin\frac{nx}{2}}{\frac{n^2}{4}}\right)\right]_0^{2\pi} = \left[\frac{-4x}{n}\cos\frac{nx}{2} + \frac{8}{n^2}\sin\frac{nx}{2}\right]_0^{2\pi}$$

$$= \frac{-8\pi}{n}\cos n\pi + \frac{8}{n^2}\sin n\pi = \frac{8\pi}{n}(-1)^{n+1} + 0, \text{ if } n=1,2,3,\ldots$$

$$= (-1)^{n+1}\frac{8\pi}{n}$$

$$F_c(n) = \int_0^l f(x)\ \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \int_0^{2\pi} 2x\cos\left(\frac{nx}{2}\right)dx = \left[2x\ \frac{\sin\frac{nx}{2}}{\frac{n}{2}}\right]_0^{2\pi} - \int_0^{2\pi} 2\cdot\frac{\sin\frac{nx}{2}}{\frac{n}{2}}\ dx$$

$$= 0 - \frac{4}{n}\int_0^{2\pi} \sin\left(\frac{nx}{2}\right)dx = \frac{4}{n}\left[\frac{\cos\frac{nx}{2}}{\frac{n}{2}}\right]_0^{2\pi}$$

$$= \frac{8}{n^2}\left[\cos n\pi - 1\right] = \frac{8}{n^2}\left[(-1)^n - 1\right]$$

3. Find the Finite Fourier cosine transform of $f(x)$ defined by $f(x) = \frac{\pi}{3} - x + \frac{x^2}{2\pi}$

Sol: $$F_c(n) = \int_0^l f(x)\ \cos\frac{n\pi x}{l}\ dx = \int_0^{\pi}\left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right)\cos nx\, dx$$

$$= \left[\left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right)\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\cdot\left(-1 + \frac{x}{\pi}\right)dx$$

$$= 0 - \frac{1}{n}\int_0^{\pi}\left(-1 + \frac{x}{\pi}\right)\sin nx\ dx$$

$$= \frac{-1}{n}\left\{\left[\left(-1 + \frac{x}{\pi}\right)\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi}\left(\frac{-\cos nx}{n}\right)\left(\frac{1}{\pi}\right)dx\right\}$$

$$= \frac{1}{n^2} - \frac{1}{n^2\pi}\int_0^{\pi}\cos nx\ dx$$

$$= \frac{1}{n^2} - \frac{1}{n^2\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = \frac{1}{n^2}, \text{ if } n > 0$$

and if $n = 0$

$$F_c(n) = \int_0^{\pi}\left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right)dx = \left[\frac{\pi}{3}x - \frac{x^2}{2} + \frac{x^3}{6\pi}\right]_0^{\pi}$$

$$= \frac{\pi^2}{3} - \frac{\pi^2}{2} + \frac{\pi^2}{6} = 0$$

4. Find the Finite Fourier Sine and cosine transforms of $f(x) = \sin ax$ in $(0,\pi)$

Sol: Finite Fourier Sine Transform:

$$F_S\{f(x)\} = F_S(n) = \int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^\pi \sin ax\cdot\sin\left(\frac{n\pi x}{\pi}\right)dx = \int_0^\pi \sin ax\cdot\sin nx\,dx$$

$$= \frac{1}{2}\int_0^\pi 2\sin ax\cdot\sin nx\,dx = \frac{1}{2}\int_0^\pi [\cos(a-n)x-\cos(a+n)x]dx$$

$$= \frac{1}{2}\int_0^\pi \left[\frac{\sin(a-n)x}{(a-n)} - \frac{\sin(a+n)x}{a+n}\right]_0^\pi$$

$$= \frac{1}{2}\left[\left(\frac{\sin(a-n)\pi}{(a-n)} - \frac{\sin(a+n)\pi}{a+n}\right) - (0-0)\right]$$

$$= \frac{1}{2}(0-0) = 0, \text{ if } a \neq n \text{ and } a,n \text{ are integers}$$

If $a=n$, then

$$F_S\{f(x)\} = F_S\{\sin ax\} = \int_0^\pi \sin^2 nx\,dx$$

$$= \frac{1}{2}\int_0^\pi (1-\cos 2nx)dx = \frac{1}{2}\left(x-\frac{\sin 2nx}{2n}\right)_0^\pi = \frac{\pi}{2}$$

$$\therefore F_S\{\sin ax\} = \begin{cases} 0, & \text{if } a\neq n \text{ and } a,n \text{ are integers} \\ \frac{\pi}{2}, & \text{if } a=n \end{cases}$$

Finite Fourier Cosine Transform:

$$F_c\{f(x)\} = F_c(n) = \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \int_0^\pi \sin ax\cdot\cos\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \int_0^\pi \sin ax\cdot\cos nx\,dx = \frac{1}{2}\int_0^\pi 2\sin ax\cdot\cos nx\,dx$$

$$= \frac{1}{2}\int_0^\pi [\sin(a+n)x + \sin(a-n)x]\,dx$$

$$= \frac{1}{2}\left[-\frac{\cos(a+n)x}{a+n} - \frac{\cos(a-n)x}{a-n}\right]_0^{\pi}$$

$$= \frac{-1}{2}\left[\left(\frac{\cos(a+n)\pi}{a+n} + \frac{\cos(a-n)\pi}{a-n}\right) - \left(\frac{1}{a+n} + \frac{1}{a-n}\right)\right]$$

$$= \frac{-1}{2}\left[\frac{(-1)^{a+n}}{a+n} + \frac{(-1)^{a-n}}{a-n} - \frac{1}{a+n} - \frac{1}{a-n}\right]$$

$$= \begin{cases} -\frac{1}{2}(0) = 0 \text{ , if } a+n \text{ is even} \\ \frac{1}{2}\left(\frac{2}{a+n} + \frac{2}{a-n}\right) = \frac{2a}{a^2-n^2} \text{ , if } a+n \text{ is odd} \end{cases}$$

$$F_c(\sin ax) = \begin{cases} 0, \text{ if } a+n \text{ is even} \\ \frac{2a}{a^2-n^2}, \text{ if } a+n \text{ is odd} \end{cases}$$

5. Find the Finite Fourier cosine Transform of $f(x) = \left(1-\frac{x}{\pi}\right)^2$ in $(0,\pi)$

Sol: $F_c(n) = F_c\{f(x)\} = \int_0^{\pi} f(x).\cos nx\,dx$

$$= \int_0^{\pi}\left(1-\frac{x}{\pi}\right)^2 \cos nx\,dx$$

$$= \left[\left(1-\frac{x}{\pi}\right)^2\left(\frac{\sin nx}{n}\right) - 2\left(1-\frac{x}{\pi}\right)\left(\frac{-1}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right) + \frac{2}{\pi^2}\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2} \text{ , if } n > 0$$

when $n=0$, $F_c(0) = \int_0^{\pi} f(x)dx = \int_0^{\pi}\left(1-\frac{x}{\pi}\right)^2 dx$

$$= \left[\frac{\left(1-\frac{x}{\pi}\right)^3}{3}(-\pi)\right]_0^{\pi} = \frac{-\pi}{3}[0-1]$$

$$= \frac{\pi}{3}$$

6. Find the Finite Fourier Sine and cosine transforms of

$$f(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ -1, & \text{if } 1 < x < 2 \end{cases}$$

Sol: Finite Fourier Sine Transform:

$$F_S\{f(x)\} = F_S(n) = \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \int_0^2 f(x) \sin nx\, dx$$

$$= \int_0^1 1\cdot \sin nx\, dx + \int_1^2 (-1) \sin nx\, dx$$

$$= \left(\frac{-\cos nx}{n}\right)_0^1 + \left(\frac{\cos nx}{n}\right)_1^2$$

$$= \frac{1}{n}\left(-\cos\frac{n}{2} + 1\right) + \frac{1}{n}\left(\cos 2n - \cos n\right)$$

$$= \frac{1}{n}\left[-\cos\left(\frac{n}{2}\right) + 1 + \cos 2n - \cos n\right]$$

$$= \frac{1}{n}\left(1 - \cos\left(\frac{n}{2}\right) + \cos 2n - \cos n\right),$$

$n = 1, 2, 3, \ldots\ldots$

Finite Fourier cosine Transform:

$$F_c\{f(x)\} = F_c(n) = \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$

$$F_c(n) = \int_0^2 f(x) \cos nx\, dx = \int_0^1 1\cdot \cos nx\, dx + \int_1^2 (-1) \cos nx\, dx$$

$$= \left(\frac{\sin nx}{n}\right)_0^1 - \left(\frac{\sin nx}{n}\right)_1^2$$

$$= \frac{1}{n}\left(\sin\frac{n}{2} - 0\right) - \frac{1}{n}\left(\sin 2n - \sin n\right)$$

$$= \frac{1}{n}\left[\sin \pi/_2 - \sin 2\pi + \sin n\right] \quad n = 1, 2, 3$$

PDE of first order UNIT-IV

formation of PDE by elemination of arbitrary constants & arbitrary functions- Solutions of 1st order linear eq (lagranges) & non nonlinear (standard type eq)

Partial DE

eq which involves Partial derivatives are called P.D.E. They must, ∴ involve atleast two independent variables & one dependent variable

Ex $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$, $x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=z^2$

Order

The order of PDE is same as that of the order of highest Partial derivative in the eq.

Ex $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$ → 1st order

Degree

The degree of an eq is the degree of highest order derivative occuring in the eq

Ex $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0$ → first degree of 2nd order

The solution of PDE is a function of independent variable, which satisfies the D.E

The general solution of a PDE contains arbitrary constants (or) arbitrary function & sometimes both

Note

whenever we consider the case of two independent variables we take them to be x & y and z to be dependent variable.

The PD Cofficients $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ will be denoted by p & q ie $\left(\boxed{\frac{\partial z}{\partial x}}\ \frac{\partial z}{\partial y}\right)$

The 2nd order PDE $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y^2}$ are denoted by r, s, t

$$\therefore \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t$$

1) Elimination of arbitrary constants

1) $z = ax + by + a^2 + b^2$

If we have $z = ax + by + a^2 + b^2$ — ①

Diff eq ① partially w.r.t x

$$\frac{\partial z}{\partial x} = a \quad ; \quad \cancel{\frac{\partial z}{\partial y} = b}$$

$$\boxed{p = a} \quad \left(\because \frac{\partial z}{\partial x} = a\right)$$

Diff eq ① partially w.r.t y

$$\frac{\partial z}{\partial y} = b$$

$$\boxed{q = b} \quad \left(\because \frac{\partial z}{\partial y} = b\right)$$

Put values a & b in eq ①

$$z = ax + by + p^2 + q^2$$

2) $z = ax + a^2y^2 + b$

Sol Gvn $z = ax + a^2y^2 + b$ - ①

Diff eq ① w.r.to x

$\frac{\partial z}{\partial x} = a \Rightarrow \boxed{p = a}$ $(\because \frac{\partial z}{\partial x} = p)$

Diff eq ① w.r.to y

$\frac{\partial z}{\partial y} = a^2(2y)$

$\boxed{q = 2a^2y}$ $(\because \frac{\partial z}{\partial y} = q)$

$\therefore \boxed{q = 2p^2y}$ which is req PDE

3) from the PDE by eliminating arbitrary Constants from $2z = (\sqrt{x+a}) + (\sqrt{y+b})$

Sol we have $2z = \sqrt{x+a} + \sqrt{y+b}$ - ①

Diff eq ① w.r.to x

$2\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x+a}}(1) + 0$ $(\because \frac{\partial z}{\partial x} = p)$

$2p = \frac{1}{2\sqrt{x+a}}$

$\boxed{\sqrt{x+a} = \frac{1}{4p}}$ - ②

Diff eq ① w.r.to y

$2\frac{\partial z}{\partial y} = 0 + \frac{1}{2\sqrt{y+b}}(1)$ $(\because \frac{\partial z}{\partial y} = q)$

$2q = \frac{1}{2\sqrt{y+b}}$

$\boxed{\sqrt{y+b} = \frac{1}{4q}}$ - ③

∴ Sub ② & ③ in eq ①

$$2z = \frac{1}{4p} + \frac{1}{4q}$$

$$2z = \frac{1}{4}\left[\frac{1}{p} + \frac{1}{q}\right] \Rightarrow 8z = \frac{p+q}{pq}$$

$$\boxed{8pqz = p+q}$$

which is req P.D.E

Elimination of Arbitrary Functions

let $u = u(x,y,z)$ & $v = v(x,y,z)$ be independent functions of the variables x, y, z and

Let $\phi(u,v) = 0$

we treat z as dependent variable & ∴ x & y independent variable

so that $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, $\frac{\partial y}{\partial x} = 0$, $\frac{\partial x}{\partial y} = 0$

1) find PDE arising from $\phi(x+y+z\,,\,x^2+y^2+z^2)=0$

Sol let $x+y+z = u$; $x^2+y^2+z^2 = v$ so that the given relation $\phi(u,v)=0$ — ①

Diff eq ① w.r.to x by chain rule

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right]+\frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right]=0 \quad ②$$

w.r.t $u = x+y+z$

$$\frac{\partial u}{\partial x} = 1 \; ; \; \frac{\partial u}{\partial z} = 1$$

$v = x^2+y^2+z^2$

$$\frac{\partial v}{\partial x} = 2x \; ; \; \frac{\partial v}{\partial z} = 2z$$

Sub above values in eq ①

$$\frac{\partial\phi}{\partial u}\left[1+1\frac{\partial z}{\partial x}\right]+\frac{\partial\phi}{\partial v}\left[2x+2z\frac{\partial z}{\partial x}\right]=0$$

$$\frac{\partial\phi}{\partial u}[1+P]+\frac{\partial\phi}{\partial v}[2x+2zP)=0 \quad \left(\because \frac{\partial z}{\partial x}=P\right) \quad ③$$

Diff eq ① w.r.t y partially

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right]+\frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right]=0 \quad ④$$

w.r.t $u = x+y+z$

$$\frac{\partial u}{\partial y} = 1 \; ; \; \frac{\partial u}{\partial z} = 1$$

$v = x^2+y^2+z^2$

$$\frac{\partial v}{\partial y} = 2y \; ; \; \frac{\partial v}{\partial z} = 2z$$

$$\frac{\partial\phi}{\partial u}\left[1+1\frac{\partial z}{\partial y}\right]+\frac{\partial\phi}{\partial v}\left[2y+2z\frac{\partial z}{\partial y}\right]=0$$

$$\frac{\partial\phi}{\partial u}[1+q)+\frac{\partial\phi}{\partial v}[2y+2zq]=0 \quad ⑤ \quad \left[\because \frac{\partial z}{\partial y}=q\right]$$

Eliminating $\frac{\partial\phi}{\partial u}$ & $\frac{\partial\phi}{\partial v}$ from ③ & ⑤ we have

$$\begin{vmatrix} 1+p & 2x+2zp \\ 1+q & 2y+2zq \end{vmatrix} = 0$$

ie $[1+p][2y+2zq] - [2x+2zp][1+q] = 0$

$2y+2zq+2py+\cancel{2zpq} - 2x - 2xq - 2zp - \cancel{2zpq} = 0$

$2y - 2x + 2p[y-z] + 2q[z-x] = 0$

$2[y - x + p(y-z) + q(z-x)] = 0$

$$\boxed{p(y-z) + q(z-x) = x-y}$$

(∵ It is in form $Pp + Qq = R$)

which is req ~~S~~ P.D.E

2) From the PDE by eliminating the arbitrary function f from $z = xy + f(x^2+y^2)$

S) we have $z = xy + f(x^2+y^2)$ — ①

Diff eq ① w.r.t to x

$\frac{\partial z}{\partial x} = \frac{\partial x}{\partial x}(y) + x(0) + f'(x^2+y^2)(2x)$ (∵ $\frac{\partial z}{\partial x} = p$)

$p = y + f'(x^2+y^2)2x$

$p - y = 2xf'(x^2+y^2$ — ②

Diff eq ① w.r.t to y

$\frac{\partial z}{\partial y} = x\frac{\partial y}{\partial y} + 0 + f'(x^2+y^2)(2y)$

$$q = x + f'(x^2+y^2)(2y)$$

$$q - x = 2y\,f'(x^2+y^2) \quad \text{—③}$$

$$\frac{①}{③} \Rightarrow \frac{p-y}{q-x} = \frac{2x f'(x^2+y^2)}{2y f'(x^2+y^2}$$

$$\frac{p-y}{q-x} = \frac{x}{y}$$

$$(p-y)y = x(q-x)$$

$$py - y^2 = qx - x^2$$

$$x^2 - y^2 = qx - py$$

which is req sol

3) from the PDE by eliminating the arbitrary function $z = f(x^2+y^2+z^2)$

Gun $z = f(x^2+y^2+z^2)$ —①

Diff eq ① w.r.t to x

$$\frac{\partial z}{\partial x} = f'(x^2+y^2+z^2)\left[(2x)+0+2z\frac{\partial z}{\partial x}\right.$$

$$p = f'(x^2+y^2+z^2)\,[2x+2zp] \quad \left(\because \frac{\partial z}{\partial x} = p\right]$$

—②

Diff eq ① w.r.t to y

$$\frac{\partial z}{\partial y} = f'(x^2+y^2+z^2)\left[2y + 2z\frac{\partial z}{\partial y}\right]$$

$$q = f'(x^2+y^2+z^2)[2y+2zq] \quad \left(\because \frac{\partial z}{\partial y} = q\right)$$

—③

$$\frac{②}{③} \Rightarrow \frac{p}{q} = \frac{\cancel{f'(x^2+y^2+z^2)}\,[2x+2zp]}{\cancel{f'(x^2+y^2+z^2]}\,[2y+2zq}$$

$q = x + f'(x^2+y^2)(2y)$

$q - x = 2y f'(x^2+y^2)$ — ③

$\frac{①}{③} \Rightarrow \frac{p-y}{q-x} = \frac{2x f'(x^2+y^2)}{2y f'(x^2+y^2}$

$\frac{p-y}{q-x} = \frac{x}{y}$

$(p-y)y = x(q-x)$

$py - y^2 = qx - x^2$

$x^2 - y^2 = qx - py$

which is req sol

3) from the PDE by eliminating the arbitrary function $z = f(x^2+y^2+z^2)$

Sol Gvn $z = f(x^2+y^2+z^2)$ — ①

Diff eq ① w.r.to x

$\frac{\partial z}{\partial x} = f'(x^2+y^2+z^2)\left[(2x)+0+2z\frac{\partial z}{\partial x}\right.$

$p = f'(x^2+y^2+z^2)[2x+2zp]$ $\left(\because \frac{\partial z}{\partial x} = p\right)$ — ②

Diff eq ① w.r.to y

$\frac{\partial z}{\partial y} = f'(x^2+y^2+z^2)\left[2y+2z\frac{\partial z}{\partial y}\right]$

$q = f'(x^2+y^2+z^2)[2y+2zq]$ — ③ $\left(\because \frac{\partial z}{\partial y} = q\right)$

$\frac{②}{③} \Rightarrow \frac{p}{q} = \frac{\cancel{f'(x^2+y^2+z^2)}[2x+2zp]}{\cancel{f'(x^2+y^2+z^2)}[2y+2zq}$

$$\frac{p}{q} = \frac{2x+2zp}{2y+2zq} \Rightarrow \frac{p}{q} = \frac{2[x+zp]}{2[y+zq]}$$

$$\frac{p}{q} = \frac{x+zp}{y+zq}$$

$$p(y+zq) = q(x+zp)$$

$$py+zpq = qx+zpq$$

$$py-qx=0 \quad (\because \text{ is in for } Pp+Qq=R)$$

This is PDE of 1st order

Linear PDE of 1st order

A DE involving partial derivatives p & q only and no higher order derivatives is called 1st order eq.

If p & q occur in 1st degree, it is called a linear P.DE of 1st order

Ex 1) $px+qy^2=z$ → is linear P.DE

2) $p^2+q^2=1$ → is non linear

Lagrange's linear Equation

A linear PDE of order one, involving a dependent variable z and two independent variables x & y of form

$$Pp+Qq=R$$

where P, Q, R are functions of x, y, z is called Lagrange's linear eq

~~General solution of linear eq~~

~~we have seen that from a relation $\phi(uv)=0$ ①~~
~~a linear Partial differential eq $Pp+Qq=R$ ②~~

is derived by eliminating the arbitrary function ϕ supp ② has been derived from ① then $\phi(u,v)=0$ is called general solution of eq ②.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

It is called Lagrange's solution of linear eq.

They are 2 types

1) Method of Grouping 2) Method of multipliers

1) Method of Grouping

In some problem, it is possible that 2 of the eq

$\frac{dx}{P} = \frac{dy}{q}$ or $\frac{dy}{Q} = \frac{dz}{R}$ or $\frac{dx}{P} = \frac{dz}{R}$

are directly soluable to get solution $u(x,y) = $ constant (or)

$v(y,z) =$ constant (or) $w(z,x) =$ constant

these give the complete solution of ①

2) Method of multipliers

This is based on the following elementary result

if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \cdots = \frac{a_n}{b_n}$ then each ratio is equal to $\frac{l_1a_1 + l_2a_2 + \cdots + l_na_n}{l_1b_1 + l_2b_2 + \cdots + l_nb_n}$

consider $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

If possible identify multipliers l, m, n not necessarily constants, so that each ratio is equal to

$$\frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

if l, m, n are so choosen that $lP + mQ + nR = 0$

then $l\,dx + m\,dy + n\,dz = 0$

Integrating this, we get $u(x,y,z)=c_1$

lly we get another solution $v(x,y,z)=c_2$

the complete sol by $u=c_1, v=c_2$

4) ✓ solve $px+qy=z$

Sol Given $px+qy=z$

It is in form $Pp+qQ=R$ ie $P=x, Q=y, R=z$

The subsidiary eq $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$$\text{ie } \frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$$

Tak

$$\frac{dx}{x}=\frac{dy}{y}$$

Integrating on both sids

$\int\frac{1}{x}dx=\int\frac{1}{y}dy$

$\log x=\log y+\log c_1$

~~$\log(x)=\log\left(\frac{y}{c_1}\right)$~~ $\log x-\log y=\log c_1$

~~$x=\frac{y}{c_1}$~~ $\log\frac{x}{y}=\log c_1$

$$\boxed{\frac{x}{y}=c_1}$$

Tak $\frac{dy}{y}=\frac{dz}{z}$

Integrating on both sids

$\int\frac{1}{y}dy=\int\frac{1}{z}dz$

$\log y=\log z+\log c_2$

$\log y-\log z=\log c_2$

$\log\left(\frac{y}{z}\right)=\log c_2$

$\Rightarrow \boxed{\frac{y}{z}=c_2}$

Hence general solution is

$f(c_1, c_2) = 0$ where f is arbitrary

ie $f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$

2) solve $y^2 p - xyq = x(z-2y)$

S Given $y^2 p - xyq = x(z-2y)$

It is in form $Pp + Qq = R$

$P = y^2$; $Q = -xy$; $R = x(z-2y)$

The eq are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Take first two we get

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$\frac{dx}{y} = \frac{dy}{-x}$ $\Rightarrow -x\,dx = y\,dy$

$\Rightarrow$ Integrating on both sides

$-\int x\,dx = \int y\,dy$

$-\frac{x^2}{2} = \frac{y^2}{2} + c_1$

$-\frac{x^2}{2} + \frac{y^2}{2} \pm c_1 = 0$

$\boxed{x^2 + y^2 = 2c_1}$ where c_1 is constant

Take last two $\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$

$$\frac{dy}{-y} = \frac{dz}{z-2y}$$

$(z-2y)dy = -y\,dz$

$z\,dy - 2y\,dy + y\,dz = 0$

$z\,dy + y\,dz - 2y\,dy = 0$

$d(yz) - 2y\,dy = 0$

Integrating above eq

$\int d(yz) - 2\int y\,dy = 0$

$yz - 2\frac{y^2}{2} = c_2$

$yz - y^2 = c_2$

Hence general sol is $f(x^2+y^2, yz-y^2) = 0$

3) Solve $(x^2-yz)p + (y^2-zx)q = z^2-xy$

Sol Given $(x^2-yz)p + (y^2-zx)q = z^2-xy$

It is in form of $Pp + Qq = R$

where $P = (x^2-yz)$; $Q = y^2-zx$; $R = z^2-xy$

The eq are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} \quad -①$$

Taking 1, -1, 0 as multipliers (we do multipliers method

$$\frac{dx - dy + dz(0)}{1(x^2-yz) - (y^2-zx) + 0(z^2-xy)} = 0$$

$$\frac{dx-dy}{x^2-yz-y^2+zx}$$

solve $x^r(y-z)p+y^2(z-x)q = z^2(x-y)$

sol The given eqn ⇒ $x^2(y-z)p+y^2(z-x)q = z^2(x-y)$

It is in form $Pp+Qq = R$

$P = x^2(y-z)$; $Q = y^2(z-x)$; $R = z^2(x-y)$

The eq are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Take $(1/x^2, 1/y^2, 1/z^2)$ as multipliers

$$\text{each fraction} = \frac{\frac{dx}{x^2}+\frac{1}{y^2}dy+\frac{1}{z^2}dz}{\frac{x^2(y-z)}{x^2}+\frac{y^2(z-x)}{y^2}+\frac{z^2(x-y)}{z^2}}$$

$$\text{each fraction} = \frac{\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2}}{y-z+z-x+x-y} \Rightarrow E.F = \frac{\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2}}{0}$$

$$\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2} = 0$$

Integrating on both sides

$$\int\frac{1}{x^2}dx+\int\frac{1}{y^2}dy+\int\frac{1}{z^2}dz = 0 \qquad \left(x^{-2} = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = \frac{-1}{x}\right)$$

$$-\frac{1}{x}-\frac{1}{y}-\frac{1}{z} = c_1$$

$$\boxed{\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = c_1}$$

Take $(1/x, 1/y, 1/z)$ as multipliers

$$\text{Each fraction} = \frac{\frac{1}{x}dx + \frac{dy}{y} + \frac{1}{z}dz}{\frac{1}{x}[x^2(y-z) + \frac{1}{y}(y^2(z-x)] + \frac{1}{z}(z^2(x-y)}$$

$$\text{Each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{x(y-z) + y(z-x) + z(x-y)}$$

$$E \cdot F = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{xy - xz + yz - yx + zx - zy}$$

$$E \cdot F = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0} \qquad (\because E \cdot F = \text{Each fraction})$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrate on both sides

$$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2$$

Hence the general sol is $\phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0$

<u>Non linear</u>

A PDE which involves first order partial derivative p & q with degree higher than one & the products of p & q is called non linear PDE

Non linear P.D.E of first order

There are 6 types of non-linear P.D.E of 1^{st} order

Type I : $f(p,q)=0$

Type II : $f(x,p,q)=0$

Type III . $f_1(x,p)=f_2(y,q)$ (variable separable)

Type IV : $z=px+qy+f(p,q)$ [clairaut form]

Type V : $f(x^m p, y^n q)=0$ and $f(x^m p, y^n q, z)=0$

Type VI : $f(pz^m, qz^m)=0$ and $f_1(x, pz^m)=f_2(y, qz^m)$

First 4 types are standard forms

Type V , type VI are reducible standard form

Type - I

Eq of form $f(p,q)=0$ (ie eq containing p&q only)

Procedure

Given Partial differential eq is of form $f(p,q)=0$ ①

Step - 1 : Put $p=a$ in eq ①

Then we get q value interms of a

Then we can obtain 'p' value

Step - 2 :- Substitute p, q value in $dz=pdx+qdy$

Step - 3 : Integrate we get req complete solution

1) Solve $pq = k$, where k is constant

~~Sol Given eq $pq = k$ —① (which is non linear eq)~~

~~Sub eq ① is of form $f(p,q)=0$~~

~~Put $p = a$ in eq ①~~

~~we get $aq = k$~~

~~$\boxed{q = \frac{k}{a}}$~~

Now substitute p, q in

Type-1

~~Step-1~~

Step-1 :- Given eq is of form $f(p,q)=0$
ie (equation containing p & q only)

Step-2 :- let the required solution be
$$z = ax+by+c \quad \text{—①}$$
where $\frac{\partial z}{\partial x} = a$; $\frac{\partial z}{\partial y} = b$

Step-3 on putting these values $f(p,q)=0$
we get $f(a,b) = 0$

Step-3 from this find the value of b
interms of a & substitute the value of b
in eq ①

Step-[illegible] then we get req solution

1) solve $p^2+q^2=1$

Sol Given eq is $p^2+q^2=1$ - ①

The eq ① is in form of $f(p,q)=0$

Then the solution of eq is

$$z=ax+by+c \quad - ②$$

where $\frac{\partial z}{\partial x}=a$; $\frac{\partial z}{\partial y}=b$

WKT $\frac{\partial z}{\partial x}=p$; $\frac{\partial z}{\partial y}=q$

$\therefore \boxed{p=a}$ & $\boxed{q=b}$

Sub p, q value in eq ①

$a^2+b^2=1 \Rightarrow b^2=1-a^2$

$\Rightarrow b=\sqrt{1-a^2}$

Put ~~these val~~ a, b vales in eq ②

$$z=ax+\sqrt{1-a^2}\,y+c$$

2) solve $pq+p+q=0$

Sol Given eq is $pq+p+q=0$ - ①

It is in form of $f(p,q)=0$

The solution of eq ① is $z=ax+by+c$ - ②

WKT $\frac{\partial z}{\partial x}=p=a$; $\frac{\partial z}{\partial y}=q=b$

Sub $p=a$, $q=b$ in eq ①

$ab+a+b=0$

~~$b=-a-ab$~~ $\Rightarrow$ ~~$b=-a[1+b]$~~

$\therefore$ ~~subst a, b values in eq ② $z=ax-a[1+b]y+c$~~

$ab + a + b = 0$

$ab + b = -a \rightarrow ⑥$

$b(1+a) = -a$

$$\boxed{b = \frac{-a}{1+a}}$$

Sub a, b values in eq ②

$$z = ax - \left(\frac{-a}{1+a}\right) y + c$$

3) Solve $3p^2 - 2q^2 = 4pq$

Given $3p^2 - 2q^2 = 4pq$ - ①

eq ① is of from $f(p,q) = 0$

Solution of eq ① is $z = ax + by + c$ - ②

$\frac{\partial z}{\partial x} = p = a$; $\frac{\partial z}{\partial y} = q = b$

Sub $p = a$ & $q = b$ values in eq ①

~~$3a^2 - 2b^2 = 4ab$~~

~~$3a^2 - 4ab + 2b^2$~~

~~$a^2 = \frac{2b^2 + 4ab}{3}$~~

~~$a = \pm\sqrt{\frac{2b^2 + 4ab}{3}}$~~

$3a^2 - 2b^2 = 4ab$

$3a^2 - 4ab = 2b^2$

$\frac{3a^2 - 4ab}{2} = b^2$

$$b = \pm\sqrt{\frac{3a^2 - 4ab}{2}}$$

Sub a, b valus in eq ①

$$z = ax \pm \sqrt{\frac{3a^2-4ab}{2}}\, y + c$$

Type-III $f(z,p,q)$ ~~$f(p,q,z)=0$~~

eq of form ~~$f(p,q,z)=0$~~ ie (equations containing p,q,z)

Step-1 eq is of form $f(z,p,q)=0$

Step-2 Let $u = x+ay$ and substitute $p=\frac{dz}{du}$ & $q = a\frac{dz}{du}$ in given eq

Step-3 Solve the resulting ODE in z & x

Step-4 Substitute x+ay for u

Ex 1 ~~Solve $z = p^2+q^2$~~

~~Sol Given eq is $z=p^2+q^2$ -①~~

~~eq ① is in form of $f(p,q,z)=0$~~

~~The solution of ① is $z=f(x+ay)$~~

~~Put $x+ay=u$ ∴ $z=f(u)$ -②~~

~~Diff eq ② w.r.to x & y we get~~

~~$\frac{dz}{du} = p$ & $q = a\frac{dz}{du}$~~ $\frac{\partial z}{\partial u} = p,\ \frac{\partial z}{\partial}$

~~Solve ODE in z & U~~ $x+ay=u$

~~$z = d^2z$~~ $\frac{\partial u}{\partial x}=1,\ \frac{\partial u}{\partial y}=a$

) Solve $p(1+q) = qz$

Given eq $p(1+q) = qz$ — ①

It is in form of $f(z, p, q) = 0$

let $u = x + ay$

$\frac{dz}{du} = p$; $q = a\frac{dz}{du}$

Sub p, q values in eq ①

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du} z$$

$$1 + a\frac{dz}{du} = az$$

$$a\frac{dz}{du} = az - 1$$

$$\frac{dz}{du} = \frac{az-1}{a}$$

$$\frac{dz}{du} = z - \frac{1}{a} \Rightarrow$$

~~Integrate on both sidy~~

$$~~\int \frac{dz}{da} =~~ \; dz = \left(z - \frac{1}{a}\right) du$$

$$\left(\frac{1}{z - \frac{1}{a}}\right) dz = du$$

Integrate on both sids

$$\int \left(\frac{1}{z - 1/a}\right) dz = \int du$$

~~$\log\left(z - \frac{1}{a}\right) = \log u + \log c$~~

$$\log\left(z - \frac{1}{a}\right) = u + c$$

$$\log\left(z - \frac{1}{a}\right) = x + ay + c \qquad (\because u = x + ay)$$

$$z - \frac{1}{a} = e^{x+ay+c}$$

✓

2) Solve $z = p^2 + q^2$

Given eq is $z = p^2 + q^2$ - ①

let $z = f(x+ay)$ be solution of eq ①

Put $x + ay = u$ then $z = f(u)$ - ②

Diff eq ② w.r.t. to x & y we get

$$\frac{dz}{du} = p \quad \& \quad a\frac{dz}{du} = q$$

sub p & q values in eq ①

$$z = \frac{d^2z}{du^2} + a^2\frac{d^2z}{du^2}$$

$$z = (1+a^2)\frac{d^2z}{du^2}$$

$$\frac{d^2z}{du^2} = \frac{z}{1+a^2}$$

$$\frac{dz}{du} = \sqrt{\frac{z}{1+a^2}} \Rightarrow dz = \frac{\sqrt{z}}{\sqrt{1+a^2}}du$$

$$\frac{dz}{\sqrt{z}} = \frac{du}{\sqrt{1+a^2}}$$ (variable seperable

Integrate on both sides

$$\int \frac{1}{\sqrt{z}}dz = \int \frac{1}{\sqrt{1+a^2}}du + c$$

$$2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}\int du + c$$

$$2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}u + c$$

$$2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}(x+ay) + c \quad (\because u = x+ay)$$

$$2\sqrt{z} = \frac{x+ay}{\sqrt{1+a^2}} + c$$

This is the req solution

3) Solve $z^2(p^2+q^2+1) = 1$

Given eq $z^2(p^2+q^2+1) = 1$ - ①

let $z =$ ~~$f(x+ay)$ be solution of eq ①~~

It is in form of $f(z, p, q) = 0$

let $z = f(x+ay)$ be the solution of eq ①

Put $u = x+ay$ then $z = f(u)$ — ②

Diff eq ② w.r.t to x & y

$\frac{dz}{du} = p$; $a\frac{dz}{du} = q$

Sub p & q values in eq ①

$$z^2\left[\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 + 1\right] = 1$$

$$z^2\left[\frac{d^2z}{du^2} + a^2\frac{d^2z}{du^2} + 1\right] = 1$$

$$\frac{d^2z}{du^2} + a^2\frac{d^2z}{du^2} + 1 = \frac{1}{z^2}$$

$$\frac{d^2z}{du^2}\left[1+a^2\right] + 1 = \frac{1}{z^2}$$

$$\frac{d^2z}{du^2}\left[1+a^2\right] = \frac{1}{z^2} - 1$$

$$\frac{d^2z}{du^2}\left[1+a^2\right] = \frac{1-z^2}{z^2}$$

$$\frac{d^2z}{du^2} = \frac{1-z^2}{z^2}\cdot\frac{1}{1+a^2}$$

$$\frac{dz}{du} = \sqrt{\frac{1-z^2}{z^2(1+a^2}}$$

$$\frac{dz}{du} = \frac{1}{z}\sqrt{\frac{1-z^2}{1+a^2}}$$

$$\frac{dz}{du} = \frac{\sqrt{1-z^2}}{z}\cdot\frac{1}{\sqrt{1+a^2}}$$

$$\frac{z}{\sqrt{1-z^2}}\,dz = \frac{1}{\sqrt{1+a^2}}\,du$$

Integrate on both side

$$\int\frac{z}{\sqrt{1-z^2}}\,dz = \int\frac{1}{\sqrt{1+a^2}}\,du \quad — ③$$

~~Put $t = z^2 - 1$~~

∴ Put $1-z^2 = t$

$$-2z\,dz = dt$$

$$z\,dz = -\frac{1}{2}dt$$

~~$\int \frac{2z}{\sqrt{1-z^2}}$~~ Sub above values in eq ③

$$\int \frac{1}{\sqrt{t}}\left(-\frac{1}{2}dt\right) = \frac{1}{\sqrt{1+a^2}}\int du + c$$

$$-\frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt = \frac{1}{\sqrt{1+a^2}}u + C$$

$$\left(\frac{1}{\sqrt{t}} = t^{-1/2}\right.$$

$$\int t^{-1/2}dt = \frac{t^{-1/2+1}}{-1/2+1}$$

$$= \frac{t^{1/2}}{1/2} = 2\sqrt{t}$$

$$-\frac{1}{2}\,2\sqrt{t} = \frac{1}{\sqrt{1+a^2}}u + c$$

$$-\sqrt{t} = \frac{1}{\sqrt{1+a^2}}u + c$$

$$-\sqrt{1-z^2} = \frac{1}{\sqrt{1+a^2}}(x+ay)+C \qquad (\because u = x+ay$$

$$\sqrt{1-z^2} = -\frac{1}{\sqrt{1+a^2}}(x+ay)+C$$

Type III (variable Seperable)

eq ~~of~~ the ~~form~~ $f_1(x,p) = f_2(y,q)$ (ie eq not involving z')

and the terms containing x & p can be seperated

from those containing y & q

where $z = \int f_1(x,a)\,dx + \int f_2(y,a)\,dy + b$

Type III (variable Seperable)

eq of form $f_1(x,P) = f_2(y,q)$

ie eq not involving z and the terms

Containing x & p can be seperated

from those containg y & q

Method of solution

~~Step-1~~

1) It is in form $f_1(x,p) = f_2(y,q) = a$ (say)

where $f_1(x,p) = a$; $f_2(y,q) = a$

2) let $p = f_1(x,a)$; $q = f_2(y,a)$

3) Solution is $dz = pdx + qdy$ $\left(\because dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad dz = pdx + qdy\right)$

$$dz = f_1(x,a)dx + f_2(y,a)dy$$

Integrate on both sides

$$\int dz = \int f_1(x,a)dx + \int f_2(y,a)dy$$

$$z = \int f_1(x,a)dx + \int f_2(y,a)dy + c$$

1) Solve $p - q = x^2 + y^2$

Given eq $p - q = x^2 + y^2$

~~eq① is in form of f(x~~

$$p - x^2 = y^2 + q = a \text{ — ①}$$

eq ① is in form of $f(x,p) = f(y,q) = a$

where $p - x^2 = a$; $y^2 + q = a$

$p = a + x^2$; $q = a - y^2$

∴ Solution is $dz = pdx + qdy$

$$dz = (a+x^2)dx + (a-y^2)dy$$

Integrate on both sides

$$\int dz = \int (a+x^2)dx + \int (a-y^2)dy$$

$$z = a\int dx + \int x^2 dx + \int a\,dy - \int y^2 dy + c$$

$$z = ax + \frac{x^3}{3} + ay - \frac{y^3}{3} + c$$

$$\boxed{z = \frac{1}{3}(x^3 - y^3 + a(x+y) + c}$$

2) solve $p+q = \sin x + \sin y$

g Given eq $p+q = \sin x + \sin y$

$$p - \sin x = \sin y - q = a \quad ①$$

eq ① is of form $f_1(x,p) = f_2(y,q) = a$

where $p - \sin x = a$ | $\sin y - q = a$

$p = a + \sin x$ | $\therefore -q = a - \sin y$

$q = \sin y - a$

∴ solution is $dz = p\,dx + q\,dy$

$$dz = (a+\sin x)dx + (\sin y - a)dy$$

Integrale above eq

$$\int dz = \int (a+\sin x)\,dx + \int (\sin y - a)\,dy$$

$$z = a\int dx + \int \sin x\,dx + \int \sin y\,dy - \int a\,dy + c$$

$$z = ax - \cos x - \cos y - ay + c$$

$$z = -\cos x - \cos y + a(x-y) + c$$

$$\boxed{\therefore z = a(x-y) - [\cos x + \cos y] + c}$$

3) Solve $\left(\frac{p}{2}+x\right)^2 + \left(\frac{q}{2}+y\right)^2 = 1$

Given eq $\left(\frac{p}{2}+x\right)^2 + \left(\frac{q}{2}+y\right)^2 = 1$

$$\left(\frac{p}{2}+x\right)^2 = 1 - \left(\frac{q}{2}+y\right)^2$$

eq ① is of form $f_1(x,p) = f_2(y,q) = a^2$ (arbitrary constant)

i.e $\left(\frac{p}{2}+x\right)^2 = 1 - \left(\frac{q}{2}+y\right)^2 = a^2$

$\left(\frac{p}{2}+x\right)^2 = a^2$

$\frac{p}{2}+x = a$

$\frac{p}{2} = a-x$

$p = 2(a-x)$

$p = 2a-2x$

$1-\left(\frac{q}{2}+y\right)^2 = a^2$

$-\left(\frac{q}{2}+y\right)^2 = a^2-1$

$\left(\frac{q}{2}+y\right)^2 = 1-a^2$

$\frac{q}{2}+y = \sqrt{1-a^2}$

$\frac{q}{2} = \sqrt{1-a^2}-y$

$q = 2\sqrt{1-a^2}-2y$

∴ Solution is $dz = pdx+qdy$

$dz = (2a-2x)dx+(2\sqrt{1-a^2}-2y)dy$

$dz = 2adx-2xdx+2\sqrt{1-a^2}dy-2ydy$

Integrate above eq on both sides

$\int dz = \int 2adx - \int 2xdx + \int 2\sqrt{1-a^2}dy - \int 2ydy +$

$z = 2a\int dx - 2\int xdx + 2\sqrt{1-a^2}\int dy - 2\int ydy + c$

$z = 2ax - \frac{2x^2}{2} + 2\sqrt{1-a^2}\,y - \frac{2y^2}{2} + c$

$$\boxed{\therefore z = 2ax - x^2 + 2\sqrt{1-a^2}\,y - y^2 + c}$$

is req complete solution

4) Solve $q = px+p^2$

Sol Given eq $q = px+p^2$ → ①

It is in form of $f(x,p) = f(y,q) = a$

$q = a$; $px+p^2 = a$

$q = a$ $\{$ $p^2 + px = a$

$p^2 + px - a = 0$ $\qquad \because x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$p = \cancel{b}\ \frac{-x \pm \sqrt{x^2 - 4(1)(-a)}}{2(1)}$

$q = a$ $\quad p = \frac{-x \pm \sqrt{x^2 + 4a}}{2}$

$\therefore dz = p\,dx + q\,dy$

$dz = \frac{-x \pm \sqrt{x^2 + 4a}}{2}\,dx + a\,dy$

~~Integ~~ $dz = -\frac{x}{2}\,dx \pm \frac{\sqrt{x^2+4a}}{2}\,dx + a\,dy$

Integrate on both sides

$\int dz = \int -\frac{x}{2}\,dx \pm \int \frac{\sqrt{x^2+4a}}{2}\,dx + a\int dy + c$

$z = -\frac{1}{2}\int x\,dx \pm \frac{1}{2}\int \sqrt{x^2+4a}\,dx + ay + c$

$z = -\frac{1}{2}\left[\frac{x^2}{2}\right] \pm \frac{1}{2}\left[\frac{x}{2}\sqrt{x^2+4a} + 2a\sinh^{-1}\frac{x}{2\sqrt{a}}\right]$
$+ ay + c$

$\because \int \sqrt{x^2 + (2\sqrt{a})^2} =$

$\boxed{\int \left(\because \sqrt{x^2+a^2} = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)\right.}$

$\therefore x = x^2 \quad a = 2\sqrt{a}$

Solve $py - q^2x^2 = x^2y$

A) Given eq $py - q^2x^2 = x^2y$ -

Dividing above eq by x^2y

$\frac{py}{x^2y} - \frac{q^2x^2}{x^2y} = \frac{x^2y}{x^2y}$

$\frac{p}{x^2} - \frac{q^2}{y} = 1 \Rightarrow \frac{p}{x^2} - 1 = \frac{q^2}{y}$ — ①

~~The eq ① becomes~~

it is in form of $f(x,p) = f(y,q) = a$

$f(x,p) = a \quad ; \quad f(y,q) = a$

ie $\frac{p}{x^2} - 1 = a \quad ; \quad \frac{q^2}{y} = a$

$\frac{p}{x^2} = a+1 \quad ; \quad q^2 = ay$

$p = x^2(a+1) \quad ; \quad q = \sqrt{a}\sqrt{y}$

∴ Solution is $dz = pdx + q\,dy$

sub p, q values

~~∴ dz =~~

$\therefore dz = x^2(a+1)dx + \sqrt{a}\sqrt{y}\,dy$

Integrate above eq

$\int dz = \int x^2(a+1)dx + \int \sqrt{a}\sqrt{y}\,dy$

$z = (a+1)\int x^2 dx + \sqrt{a}\int \sqrt{y}\,dy + c$

$z = (a+1)\frac{x^3}{3} + \sqrt{a}\,\frac{2}{3}y^{3/2} + c$

$\sqrt{y} = y^{1/2}$

$\int\sqrt{y} = \int y^{1/2}dy$

$= \frac{y^{1/2+1}}{1/2+1} = \frac{y^{3/}}{3/}$

which is complete integral

Standard form IV

eq of the form $z = px + qy + f(p,q)$

An eq analogous to clairaut's ODE

$y = px + f(p$

The complete Solution of eq a complete integral of eq

$z = px + qy + f(p,q)$

$z = ax + by + f(a,b)$ — ① $(\because p = a, q = b)$

let req solution be $z = ax + by + c$

then $\frac{\partial z}{\partial a} = 0$, $\frac{\partial z}{\partial b} = 0$ are singular soln

$\boxed{p = \frac{\partial z}{\partial x} = a}$ & $\boxed{q = \frac{\partial z}{\partial y} = b}$ (x, y & q, z by standard form of only power 1 that

Put a & b vals in eq ①

then

$$z = px + qy + f(p,q) \quad (\because p = a, q = b)$$

Ex ✓ $z = px + qy + pq$

S Given eq $z = px + qy + pq$ — ①

eq ① is in form of $z = px + qy + f(p,q)$

Hence complete solution is given by ~~z = ax + by + ab — ②~~

$z = ax + by + ab$ — ②

Diff eq ② w.r.t to a

$\frac{\partial z}{\partial a} = \frac{\partial}{\partial a}(ax + by + ab) \Rightarrow 0 = \frac{\partial}{\partial a}(ax) + \frac{\partial}{\partial a}(by) + \frac{\partial}{\partial a}(ab)$

$0 = x + b \rightarrow \cancel{③} \Rightarrow \boxed{b = -x}$ — ③

Diff eq ② w.r.t to b

$\frac{\partial z}{\partial b} = \frac{\partial}{\partial b}(ax + by + ab)$

$0 = y + a \rightarrow \cancel{④}$ $\boxed{a = -y}$ — ④

Sub ③ & ④ values in eq ②

$z = -yx - xy + xy$

$\boxed{z = -xy}$

which is singular Solution

2) Prove that complete integral of $z = px + qy + \sqrt{p^2 + q^2 + 1}$ represents all plane at unit distance from the origin

S Given eq $z = px + qy + \sqrt{p^2 + q^2 + 1}$ — ①

The eq ① is of form of $z = px+qy+f(p,q)$

∴ complete solution is $z = ax+by+\sqrt{a^2+b^2+1}$ $(\because p=a, q=b)$

or $ax+by-z+\sqrt{a^2+b^2+1} = 0$ — ③

which is a family of planes

The length of perpendicular drawn from the origin to the plane -② eq is $\frac{\sqrt{1+a^2+b^2}}{\sqrt{1+a^2+b^2}} = 1$

3) Solve $(p+q)(z-px-qy) = 1$

A Given eq is $(p+q)(z-px-qy) = 1$

$$z-px-qy = \frac{1}{p+q}$$

$$z = px+qy+\frac{1}{p+q} \quad -①$$

eq ① is in form $z = px+qy+f(p,q)$

then complete solution is $z = ax+by+\frac{1}{a+b}$

4) Solve $z = px+qy+p^2q^2$

A Given eq $z = px+qy+p^2q^2$ —①

It is in form of $z = px+qy+f(p,q)$

complete solution $z = ax+by+a^2b^2$ —② $(\because p=a, q=b)$

Diff eq ② w.r.t to a

$$\frac{\partial z}{\partial a} = \frac{\partial}{\partial a}(ax+by+a^2b^2)$$

$$0 = x+2ab^2$$

$$\boxed{x = -2ab^2} \quad — ③$$

Diff eq ② w.r.t to b

$$\frac{\partial z}{\partial b} = \frac{\partial}{\partial b}(ax+by+a^2b^2)$$

$0 = 0 + y + a^2(2b)$

$\boxed{y = -2a^2b}$ - ④

from ③ $x = -2ab^2$

$\frac{x}{b} = -2ab$

from ④ $\Rightarrow$ $y = -2a^2b$

$\frac{y}{a} = -2ab$

$\therefore \frac{x}{b} = \frac{y}{a} = -2ab = \frac{1}{k}$ (say)

$\frac{x}{b} = \frac{1}{k}$ | $\frac{y}{a} = 1/k$

$b = kx$ | $a = ky$

Sub a, b values in eq ③

$x = -2(ky)(kx)^2$

$x = -2ky(k^2x^2)$

$x = -2k^3yx^2$

$k^3 = \frac{x}{-2yx^2}$

$k^3 = -\frac{1}{2yx}$

Sub a, b values in eq ②.

$z = (ky)x + (kx)y + (ky)^2(kx)^2$

$z = kxy + kxy + (k^2y^2k^2x^2)$

$z = 2kxy + k^4x^2y^2$

$z = 2kxy + k^3(kx^2y^2)$ $\left(\because k^3 = -\frac{1}{2xy}\right)$

$z = 2kxy \; \frac{1}{2yx}(kx^2y^2)$

$z = 2kxy - \frac{kxy}{2} \Rightarrow z = \frac{4kxy - kxy}{2}$

$z = \frac{3kxy}{2}$

Equation reducible to standard form

Type V & Type VI

eq. of the type $f(x^m p, y^n q) = 0$ where m & n are constant

The above eq can be transformed to an eq of the form $f(P, Q) = 0$ by sub given below

Case-I

when $m \neq 1$ & $n \neq 1$

Put $X = x^{1-m}$ & $Y = y^{1-n}$ then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = P(1-m)x^{-m} \quad \text{where } P = \frac{\partial z}{\partial X}$$

$$\left(\because X = x^{1-m} \quad \frac{\partial X}{\partial x} = (1-m)x^{(1-m)-1} = (1-m)x^{-m}\right.$$

$$\Rightarrow x^m p = P(1-m)$$

$$\& \quad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y} = Q(1-n)y^{-n} \quad \text{where } Q = \frac{\partial z}{\partial Y}$$

$$\Rightarrow y^n q = Q(1-n)$$

Now the given eq reduces to $f[(1-m)P, (1-n)Q] = 0$

which is of form $f(P, Q) = 0$

Case II

when $m = 1$, $n = 1$

Put $X = \log x$ & $Y = \log y$ then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{1}{x} \qquad \left(X = \log x \quad \frac{\partial X}{\partial x} = \frac{1}{x}\right)$$

~~$P =$~~ $p = \frac{\partial z}{\partial X}\cdot\frac{1}{x}$

$$px = P \qquad \left(\because P = \frac{\partial z}{\partial x}\right.$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y}$$

$$q = \frac{\partial z}{\partial Y}\cdot\frac{1}{y} \qquad \left(\because Y = \log y \quad \frac{\partial Y}{\partial y} = 1/y\right)$$

$$qy = Q \qquad \left(\because Q = \frac{\partial z}{\partial y}\right)$$

Now given eq reduces to form $f(P, Q) = 0$

) Equations of the type $f(x^m p, y^n q, z) = 0$ where m kn are constants

This can be reduced of form $f(P, Q, z) = 0$ by substitution given for eq $f(x^m p, y^n q) = 0$

Type-II

3) Equations of type $f(pz^n, qz^n) = 0$ where n is constat

use following substitution to reduce to above form to an eq of form $f(P, Q) = 0$

Put $Z = \begin{cases} z^{n+1}, & \text{if } n \neq -1 \\ \log z, & \text{if } n = -1 \end{cases}$

2) Equation of type $f(x, pz^n) = g(y, qz^n)$ where n is constant

An eq of above form can be reduced to an eq of the form $f(P, Q) = 0$ by substitutes given for eq $f(p \cdot z^n, qz^n) = 0$ as above

Ex! Solve $xp + yq = 1$

Sol Given eq $xp + yq = 1$ ①

eq ① is of form $f(x^m p, y^n q) = 0$

Here $m = 1$, $n = 1$

$\therefore$ Put $\log x = X$ and $\log y = Y$

$(\because m = 1, n = 1$
$xp, yq = 0$
$\Rightarrow xp + yq = 1$

Now $p = \frac{\partial z}{\partial x}$

$p = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x}$

$p = P \cdot \frac{1}{x}$ $\quad (\because \; X = \log x \; ; \; \frac{\partial X}{\partial x} = \frac{1}{x} \; ; \; \frac{\partial z}{\partial X} = P]$

$xp = P$

$\therefore \boxed{P = px}$ -②

$q = \frac{\partial z}{\partial y}$

$q = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y}$ $\quad$ $\left(\because Y = \log y \right.$

$\frac{\partial Y}{\partial y} = \frac{1}{y}$

$q = Q \cdot \frac{1}{y}$

$\boxed{Q = qy}$ – ③ $\quad$ $\left. \because \frac{\partial z}{\partial Y} = Q \right)$

Sub ② & ③ in eq ① then

$P + Q = 1$ – ④

let $z = f(u)$ where

$u = X + aY$ be solution of ④

(∵ In type II
this is for
$f(z,P,Q)=0$
$z = f(u)$
$u = x + ay$
$\frac{\partial z}{\partial u} = P$, eq=ad

then $P = \frac{dz}{du}$; $Q = a\frac{dz}{du}$

Sub P, Q value in eq ④

$$\frac{dz}{du} + a\frac{dz}{du} = 1$$

$$\frac{dz}{du}[1+a] = 1$$

$$\frac{dz}{du} = \frac{1}{1+a}$$

$$dz = \frac{1}{1+a}du$$

Integrating above eq

$$\int dz = \int \frac{1}{1+a} du + c$$

$$z = \frac{1}{1+a} u + c$$

$$z = \frac{1}{1+a}(X + aY) + c \quad (\because u = X + aY)$$

∴ The general sol is

$$z = \frac{1}{1+a}[\log x + a\log y] + c$$

✓ Solve $z^2 = x^2p^2 + y^2q^2$

Given eq $z^2 = (xp)^2 + (yq)^2$ —①

eq ① is of form $f(z, x^m p, y^n q) = 0$

then $m = 1, n = 1$

∴ Put $X = \log x$

$P = \frac{\partial z}{\partial X}$

$p = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x}$

$= P \frac{\partial}{\partial x} (\log x$

$p = P \frac{1}{x}$

$\boxed{P = px}$

put $Y = \log y$ then

$Q = \frac{\partial z}{\partial Y}$

$q = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y}$

$= Q \cdot \frac{\partial}{\partial y} (\log y)$

$q = Q \frac{1}{y}$

$\boxed{Q = qy}$

Sub P, Q values in eq ①

$z^2 = P^2 + Q^2$ — ②

This is in form $f(z, P, Q) = 0$

let $z = f(u)$ where $u = X + aY$ be solution of ②

then $P = \frac{dz}{du}$ & $Q = a\frac{dz}{du}$ — ③

Sub ③ in eq ②

$$z^2 = \left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2$$

$$z^2 = \left(\frac{dz}{du}\right)^2 [1 + a^2] \Rightarrow \frac{z^2}{1+a^2} = \left(\frac{dz}{du}\right)^2$$

$$\frac{dz}{du} = \sqrt{\frac{z^2}{1+a^2}}$$

$$\frac{dz}{du} = \frac{z}{\sqrt{1+a^2}}$$

$$\frac{1}{z} dz = \frac{1}{\sqrt{1+a^2}} du$$

Integrate on both sides

$$\int \frac{1}{z} dz = \frac{1}{\sqrt{1+a^2}} \int du + c$$

$$\log z = \frac{u}{\sqrt{1+a^2}} + c$$

∴ Complete integral is

$$\log z = \frac{x + aY}{\sqrt{1+a^2}} + c$$

$$\log z = \frac{\log x + a\log y}{\sqrt{1+a^2}} + c$$

3) Solve $\frac{x^2}{p} + \frac{y^2}{q} = z$

Sol Given eq $\frac{x^2}{p} + \frac{y^2}{q} = z$

$$x^2 p^{-1} + y^2 q^{-1} = z$$

$$(x^{-2}p)^{-1} + (y^{-2}q)^{-1} = z \quad \text{—①}$$

eq① is in form of $f(x^m p, y^n q, z) = 0$

with $m = -2$, $n = -2$

∴ Put $X = x^{1-m}$

$$X = x^{1-(-2)} = x^{1+2} = x^3$$

$$\therefore X = x^3$$

Now $p = \frac{\partial z}{\partial x}$

$$p = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x}$$

$$p = P\frac{\partial}{\partial x}(x^3)$$

$$p = P3x^2$$

$$\frac{p}{x^2} = 3P$$

$$3P = px^{-2}$$

$$Y = y^{1-m}$$

$$Y = y^{1-(-2)} = y^{1+2} = y^3$$

$$Y = y^3$$

Now $q = \frac{\partial z}{\partial y}$

$$q = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y}$$

$$q = Q \cdot \frac{\partial}{\partial y}(y^3)$$

$$q = Q\,3y^2$$

$$\frac{q}{y^2} = 3Q$$

$$3Q = qy^{-2}$$

Sub px^{-2}, qy^{-2} values in eq ①

$$\therefore (3P)^{-1} + (3Q)^{-1} = z \quad \text{—②}$$

eq ② is in form of $f(z, P, Q) = 0$

let $z = f(u)$ were $u = x + ay$ be solution of ③

$P = \frac{dz}{du}$, $Q = a\frac{dz}{du}$

sab P Q value ies eq ②

$$\left(3\frac{dz}{du}\right)^{-1} + \left(3a\,\frac{dz}{du}\right)^{-1} = z$$

$$3^{-1}\frac{du}{dz} + (3a)^{-1}\frac{du}{dz} = z$$

$$\frac{1}{3}\frac{du}{dz} + \frac{1}{3a}\frac{du}{dz} = x$$

$$\frac{1}{3}\frac{du}{dz}\left[1 + \frac{1}{a}\right] = z$$

$$z\,dz = \left(\frac{a+1}{3a}\right)du$$

Integrale above eq

$$\int z\,dz = \frac{a+1}{3a}\int du + c$$

$$\frac{z^2}{2} = \frac{a+1}{3a}u + c$$

$$\frac{z^2}{2} = \left(\frac{a+1}{3a}\right)(x + ay) + c \qquad (u = x + ay)$$

$$\qquad (\because X = x^3 ; Y = y^3)$$

$$\frac{z^2}{2} = \left(\frac{a+1}{3a}\right)(x^3 + ay^3) + c$$

4) Sdue ~~$x^2p^2 + y^2q^2 = z^2$~~

~~Sl Gven~~ eq . 6th modu

4) Solve $(zp + x)^2 + (zq + y)^2 = 1$

Sl Gen eq $(zp + x)^2 + (zq + y)^2 = 1$ – ①

eq ① is of form $f_1(x, z^m p) = f_2(y, z^m q)$

Here $m = 1$

Put $z^{m+1} = Z$

$z^{1+1} = Z \Rightarrow \boxed{z^2 = Z}$

~~Now $P = \frac{\partial z}{\partial x} \Rightarrow P = \partial z$~~

$P = \frac{\partial Z}{\partial x}$

$P = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x}$

~~$\frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x}$~~

$P = \frac{\partial}{\partial z}[z^2] \cdot p$

$P = 2zp$

$\boxed{zp = \frac{P}{2}}$

$Q = \frac{\partial Z}{\partial y}$

$Q = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y}$

$Q = \frac{\partial}{\partial z}[z^2] \cdot q$

$Q = 2zq$

$\boxed{zq = \frac{Q}{2}}$

Sub zp, zq value in eq ①

$\left(\frac{P}{2}+x\right)^2 + \left(\frac{Q}{2}+y\right]^2 = 1$ — ②

~~eq ② is of form $f_1(x,P) = f_2(y,Q)$~~

$\left(\frac{P}{2}+x\right)^2 = 1-\left(\frac{Q}{2}+y\right)^2$ — ③

eq ③ is of form $f_1(x,P) = f_2(y,Q)$

let $f_1(x,P) = a^2$; $f_2(y,Q) = a^2$ (say)

$\left(\frac{P}{2}+x\right)^2 = a^2$; $1-\left(\frac{Q}{2}+y\right)^2 = a^2$

$\frac{P}{2}+x = a$; $-\left(\frac{Q}{2}+y\right)^2 = a^2-1$

$\frac{P+2x}{2} = a$; $\left(\frac{Q}{2}+y\right)^2 = 1-a^2$

$P+2x = 2a$; $\frac{Q}{2}+y = \sqrt{1-a^2}$

$P = 2a-2x$; $\frac{Q}{2} = \sqrt{1-a^2}-y$

$P = 2(a-x)$; $Q = 2\left[\sqrt{1-a^2}-y\right]$

wkt $dZ = Pdx + Qdy$

$$dZ = 2(a-x)dx + 2[\sqrt{1-a^2} - y]dy$$

Integrate above eq

$$\int dZ = 2\int \cancel{2}(a-x)dx + 2\int[\sqrt{1-a^2} - y]dy + c$$

$$Z = 2\left[\int a\,dx - \int x\,dx\right] + 2\left[\int \sqrt{1-a^2}\,dy - \int y\,dy\right] + c$$

$$Z = 2\left[ax - \frac{x^2}{2}\right] + 2\left[(\sqrt{1-a^2})y - \frac{y^2}{2}\right] + c$$

$$Z = 2ax - x^2 + 2y\sqrt{1-a^2} - y^2 + c$$

∴ The general solution is

$$z^2 = 2ax - x^2 + 2y\sqrt{1-a^2} - y^2 + c$$

✓ Solve $4z^2q^2 = y + 2zp - x$ → 6^{th} model

Gen eq $4z^2q^2 = y + 2zp - x$

$$4z^2q^2 - y = 2zp - x$$

$$4(zq)^2 - y = 2(zp) - x \quad \text{—} ①$$

eq ① is of form $f_1(x, z^m p) = f_2(y, z^m q)$

∴ Put $z^{m+1} \overset{\underline{m=1}}{=} Z$ (capital z)

$$z^{1+1} = z^2 = Z$$

ie $\boxed{Z = z^2}$

now $P = \frac{\partial Z}{\partial x}$

$$P = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$P = \frac{\partial}{\partial z}(z^2) \cdot p$$

$$P = 2zp$$

$$\boxed{zp = P/2}$$

$Q = \frac{\partial Z}{\partial y}$

$$Q = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y}$$

$$Q = \frac{\partial}{\partial z}(z^2) \cdot q$$

$$Q = 2zq$$

$$\boxed{zq = Q/2}$$

Sub zp, zq valus in eq ①

$$4\left(\frac{Q}{2}\right)^2 - y = 2\left(\frac{P}{2}\right) - a$$

ie $4\frac{Q^2}{4} - y = P - x$

$$Q^2 - y = P - x \quad - ③$$

eq ③ is of form $f_1(x, P) = f_2(y, Q)$

ie $f_1(x, P) = f_2(y, Q) = a$

let $Q^2 - y = a$ | $P - x = a$

$Q^2 = a + y$ | $P = x + a$

$Q = \sqrt{a+y}$ | $P = a + x$

~~Sub P, Q vals~~

wkt $dZ = Pdx + Qdy$

$$dZ = (a+x)dx + \sqrt{a+y}\,dy$$

Integrate above eq

$$\int dZ = \int (a+x)dx + \int \sqrt{a+y}\,dy$$

$$Z = ax + \frac{x^2}{2} + \frac{2}{3}(\sqrt{a+y})^{3/2} + c$$

General solution is

$$z^2 = ax + \frac{x^2}{2} + \frac{2}{3}(a+y)^{3/2} + c$$

① M-III Unit - V

Higher order partial Differential Equations

Partial Differential equ'n :- A D.E which involves partial derivatives is called a partial differential equ'n

Ex :- (i) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$ (ii) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ are P.D.E

Order of P.D.E The order of the P.D.E is the highest ordered partial derivative in the equation.

Degree of P.D.E The Degree of The P.D.E is the highest power of the highest ordered derivative.

Ex :- $\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial z}\right)^3$ order = 2 Degree = 1

If degree of P.D.E is one then it is called Linear P.D.E.

Note If z is a function of two independent variables x and y, then we shall use the following notation for the partial derivatives of z.

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

In this chapter we shall consider only Linear partial differential equation of higher order with Constant Coefficients. It can be divided into two parts

1. **Homogeneous Linear P.D.E** (each term order is same, degree is one)

Ex :- $\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial^3 u}{\partial x^2 \partial y} + 4\frac{\partial^3 u}{\partial y^2} = \log(x+2y)$

2. **Non Homogeneous Linear P.D.E** (Having different order terms, degree is one)

Ex :- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 3\frac{\partial u}{\partial x} - 4\frac{\partial u}{\partial y} = \sin(x+y)$

Homogeneous Linear Partial differential equ'n with Constant Coefficients :- An equ'n of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = Q(x,y) \quad \text{—①}$$

②

where $a_0, a_1, a_2, \ldots, a_n$ are Constants, is called a homogeneous linear partial differential equation with constant Coefficients of n^{th} order.

Here all the partial derivatives are of the n^{th} order.

Here we take $\boxed{\frac{\partial}{\partial x} = D, \quad \frac{\partial}{\partial y} = D^1}$

∴ The symbolic form of eqn ① is

$$(a_0 D^n + a_1 D^{n-1} D^1 + a_2 D^{n-2} D^{12} + \cdots + a_n D^{n}) z = Q(x,y)$$

$$\text{ie,} \quad f(D, D^1) z = Q(x,y))$$

Its soln consists of two parts, The complementary soln (C.F) and the particular integral (P.I)

The complementary function is the general soln of $f(D, D^1) = 0$. it must ~~Const~~ Contain 'n' arbitrary Constants where n is the order of The P.D.E.

The particular integral is a particular soln (free from arbitrary Constants) of $f(D, D^1) = Q(x,y)$.

∴ The Complete soln of eqn ① is $\boxed{z = C.F + P.I}$

<u>Rules to find Complementary function.</u>

putting $D = m$ and $D^1 = 1$ in $f(D, D^1) = 0$, we get The auxillary eqn (A.E) $f(m, 1) = 0$, solve it for m

(i) If the roots of A.E are $m_1, m_2, \ldots, m_n$ all are destinct (real or imaginary) then

$$C.F = f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + \cdots + f_n(y + m_n x)$$

(ii) If two roots are equal and remaining roots are different ie $m_1 = m_2 = m$, $m_3, m_4, \ldots, m_n$ then

$$C.F = f_1(y + mx) + x f_2(y + mx) + f_3(y + m_3 x) + f_4(y + m_4 x) + \cdots + f_n(y + m_n x)$$

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③ (iii) If Three roots are equal and remaining roots are different i.e $m_1 = m_2 = m_3 = m$, $m_4, m_5, \dots, m_n$

Then $C.F = f_1(y+mx) + x f_2(y+mx) + x^2 f_3(y+mx) + f_4(y+m_4x) + f_5(y+m_5x) + \dots + f_n(y+m_nx)$

where $f_1, f_2, f_3, \dots, f_n$ are arbitrary functions.

Problems.

① Solve $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$.

Sol:- Since in the given P.D.E each term is same order so this is Homogeneous P.D.E.

Symbolic form of the given eqn is

$$(2D^2 + 5DD' + 2D'^2)z = 0 \quad \left(\text{put } \frac{\partial}{\partial x} = D, \frac{\partial}{\partial y} = D'\right)$$

Clearly This is of the form $f(D,D')z = 0$.

where $f(D,D') = 2D^2 + 5DD' + 2D'^2$

To get the auxillary eqn (A.E) put $D = m, D' = 1$

we have $f(m,1) = 0$.

$$2m^2 + 5m + 2 = 0.$$

$$\Rightarrow (2m+1)(m+2) = 0$$

$\Rightarrow m_1 = -\frac{1}{2}, m_2 = -2$ are different roots

∴ General soln $z = C.F$

$$= f_1(y+m_1x) + f_2(y+m_2x)$$

$$z = f_1(y - \tfrac{1}{2}x) + f_2(y-2x).$$

② Solve $4r + 12s + 9t = 0$.

W.K.T $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

∴ Given P.D.E is $4\frac{\partial^2 z}{\partial x^2} + 12\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0$.

Symbolic form is $(4D^2 + 12DD' + 9D'^2)z = 0$.

④ Here $f(D,D') = 4D^2+12DD'+9D'^2$

A·E is $f(m,1)=0$

$\Rightarrow 4m^2+12m+9=0$

$\Rightarrow m = \frac{-12 \pm \sqrt{12^2-4\times4\times9}}{2\times4}$ $\left(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right.$

$\left. a=4, b=12, c=9\right)$

$\Rightarrow m = \frac{-12\pm\sqrt{144-144}}{8}$

$\Rightarrow m = \frac{-3}{2}$

$\therefore m = -\frac{3}{2}, -\frac{3}{2}$ are equal roots

$C\cdot F = f_1(y+mx)+x\, f_2(y+mx)$

$= f_1(y-\frac{3}{2}x) + x\, f_2(y-\frac{3}{2}x)$ or

$C\cdot F = f_1(2y-3x)+x\, f_2(2y-3x)$

Since R·H·S of the given eq'n is zero, The complete soln is $z = C\cdot F$

$$\Rightarrow z = f_1(2y-3x)+x\, f_2(2y-3x)$$

③ Solve $(D^3-4D^2D'+4DD'^2)z=0$.

Sol:- Here $f(D,D') = D^3-4D^2D'+4DD'^2$

A·E is $f(m,1)=0$ (ie put $D=m, D'=1$)

$m^3-4m^2+4m=0$

$\Rightarrow m(m^2-4m+4)=0$

$\Rightarrow m(m-2)^2=0$

$\Rightarrow m=0,2,2$ ie $m_1=0$, $m_2=m_3=2$

Here one root is different and remaining two roots are equal roots

$\therefore C\cdot F = f_1(y+m_1x)+f_2(y+m_2x)+x\, f_3(y+m_2x)$

$\Rightarrow C\cdot F = f_1(y)+f_2(y+2x)+x\, f_3(y+2x)$

General soln is $z=C\cdot F = f_1(y)+f_2(y+2x)+x\, f_3(y+2x)$.

⑤

④ Solve $\frac{\partial^3 z}{\partial x^3} + 2\frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} - 2\frac{\partial^3 z}{\partial y^3} = 0.$

Sol:— Symbolic form of the given Homo P.D.E is

$(D^3 + 2D^2D' - DD'^2 - 2D'^3)z = 0.$

Here $f(D,D') = D^3 + 2D^2D' - DD'^2 - 2D'^3$

A.E is $f(m,1) = 0$

$\Rightarrow m^3 + 2m^2 - m - 2 = 0$

$m_1 = 1,\ m_2 = -1,\ m_3 = -2$

are different roots

$$\begin{array}{c|cccc}
1 & 1 & 2 & -1 & -2 \\
 & 0 & 1 & 3 & 2 \\ \hline
-1 & 1 & 3 & 2 & \boxed{0} \\
 & 0 & -1 & -2 & \\ \hline
-2 & 1 & 2 & \boxed{0} & \\
 & 0 & -2 & & \\ \hline
 & 1 & \boxed{0} & &
\end{array}$$

∴ G.S is $z = C.F$

$z = f_1(y+m_1x) + f_2(y+m_2x) + f_3(y+m_3x)$

$\Rightarrow z = f_1(y+x) + f_2(y-x) + f_3(y-2x).$

⑤ Solve $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0.$

Sol:— Symbolic form is $(D^4 - D'^4)z = 0$

Here $f(D,D') = D^4 - D'^4$

A.E is $f(m,1) = 0$

$\Rightarrow m^4 - 1 = 0$

$\Rightarrow (m^2+1)(m^2-1) = 0$

$\Rightarrow m^2+1 = 0,\ m^2-1 = 0$

$\Rightarrow m^2 = -1,\ m^2 = 1$

$\Rightarrow m = \pm i,\ m^2 = \pm 1$

∴ roots are $m_1 = -i,\ m_2 = i,\ m_3 = -1,\ m_4 = 1$ are different roots

$z = C.F$

General soln is $z = f_1(y+m_1x) + f_2(y+m_2x) + f_3(y+m_3x) + f_4(y+m_4x)$

$\Rightarrow z = f_1(y-ix) + f_2(y+ix) + f_3(y-x) + f_4(y+x)$

⑥

⑥ $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 3\frac{\partial^3 z}{\partial x \partial y^2} = 0$

Sol:- symbolic folm of the given Homo P.D.E is

$(D^3 - 4D^2D' + 3DD'^2)z = 0$.

A.E is $f(m, 1) = 0$.

$\Rightarrow m^3 - 4m^2 + 3m = 0$.

$\Rightarrow m(m^2 - 4m + 3) = 0$

$\Rightarrow m(m-1)(m-3) = 0$

$\Rightarrow m_1 = 0, m_2 = 1, m_3 = 3$ are different roots

$\therefore$ G.S is $z = C.F = f_1(y+m_1x) + f_2(y+m_2x) + f_3(y+m_3x)$

$\Rightarrow z = f_1(y) + f_2(y+x) + f_3(y+3x)$.

H.W

① $\frac{\partial^3 z}{\partial x^3} - 7\frac{\partial^3 z}{\partial x \partial y^2} + 6\frac{\partial^3 z}{\partial y^3} = 0$, roots are $m = 1, 2, -3$

② $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$, roots are $m = 1, 2, 3$.

③ $(D + 2D')(D - 3D')^2 z = 0$ roots are $m = -2, 3, 3$.

⑦ $\frac{\partial^4 z}{\partial x^4} - 2\frac{\partial^4 z}{\partial x^3 \partial y} + 2\frac{\partial^4 z}{\partial x \partial y^3} - \frac{\partial^4 z}{\partial y^4} = 0$

Sol:- symbolic folm of given Homo P.D.E is

$(D^4 - 2D^3D' + 2DD'^3 - D'^4)z = 0$. (put $\frac{\partial}{\partial x} = D, \frac{\partial}{\partial y} = D'$)

Here $f(D, D') = D^4 - 2D^3D' + 2DD'^3 - D'^4$

A.E is $f(m, 1) = 0$ (Put $D = m, D' = 1$)

$\Rightarrow m^4 - 2m^3 + 2m - 1 = 0$

$$\cancel{\begin{array}{r|rrr|r} 1 & 1 & -2 & 2 & -1 \\ & 0 & 1 & -1 & 1 \\ \hline & 1 & -1 & 1 & 0 \end{array}}$$

$$\begin{array}{r|rrrr|r} 1 & 1 & -2 & 0 & 2 & -1 \\ & 0 & 1 & -1 & -1 & 1 \\ \hline 1 & 1 & -1 & -1 & 1 & 0 \\ & 0 & 1 & 0 & -1 & \\ \hline 1 & 1 & 0 & -1 & 0 & \\ & 0 & 1 & 1 & & \\ \hline -1 & 1 & 1 & 0 & & \\ & 0 & -1 & & & \\ \hline & 1 & 0 & & & \end{array}$$

G.S is $z = C.F$

$\Rightarrow z = f_1(y+m_1x) + f_2(y+m_2x) + x f_3(y+m_2x) + x^2 f_4(y+m_2x)$

$\Rightarrow z = f_1(y-x) + f_2(y+x) + x f_3(y+x) + x^2 f_4(y+x)$

roots are $m_1 = -1$, $m_2 = m_3 = m_4 = 1$

ie one root is different, remainiy 3roots are equl

(7)

Rules for finding P.I

particular integral of the eqn $f(D,D')z = Q(x,y)$ is

$$z_p = \frac{1}{f(D,D')} Q(x,y)$$

where $Q(x,y)$ is a function of $ax+by$

i.e, e^{ax+by}, $\sin(ax+by)$, $\cos(ax+by)$, $\tan(ax+by)$,

$(ax+by)^n$, $\log(ax+by)$ etc Then

Case(i)

$$P.I = \frac{1}{f(D,D')} Q(ax+by)$$

$$= \frac{1}{f(a,b)} \underbrace{\iiint\cdots\int}_{(n\text{ times})} Q(u)\, \underbrace{du\,du\cdots du}_{(n\text{ times})}$$

provided $f(a,b) \neq 0$

ie ① replace D by a, D' by b where a,b are coefficients of x and y respectively in $ax+by$.

This method is applicable only when $f(a,b) \neq 0$.

If $f(a,b)=0$, then this method fails

② If $f(a,b) \neq 0$, put $ax+by=u$; in $Q(ax+by)$ to get $Q(u)$, integrate $Q(u)$ ntimes ie 'n' times

ie as many times as the degree of the P.D.E (order of P.D.E)

③ Now replace u by $ax+by$ to get the required P.I

Case(ii) Case of failure ie $f(a,b)=0$

Then $$P.I = x.\frac{1}{\frac{\partial}{\partial D}[f(D,D')]} Q(ax+by)$$

ie multiply by x and differentiate $f(D,D')$ partially w.r.to D

If the method fails again, multiply again by x and differentiate the denominator partially w.r.to D.

⑧ Problems

① solve $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$

clearly given P.D.E is Homogeneous since each term order is same.

Symbolic form of The given P.D.E is

$$\left(D^3 - 3D^2D' + 4D'^3\right) z = e^{x+2y}$$

clearly This is of The form $f(D,D')z = Q(x,y)$

$f(D,D') = D^3 - 3D^2D' + 4D'^3$, $Q(x,y) = e^{x+2y}$

A.E is $f(m,1) = 0$

$\Rightarrow m^3 - 3m^2 + 4 = 0$

$m_1 = -1,\ m_2 = m_3 = 2 = m$

Here one root is different root and remaining two roots are equal roots

$$\begin{array}{r|rrr|r}
-1 & 1 & -3 & 0 & 4 \\
 & 0 & -1 & +4 & -4 \\
\hline
2 & 1 & -4 & +4 & 0 \\
 & 0 & 2 & -4 & \\
\hline
2 & 1 & -2 & 0 & \\
 & 0 & 2 & & \\
\hline
 & 1 & 0 & &
\end{array}$$

$\therefore$ C.F = $f_1(y+m_1x) + f_2(y+mx) + x f_3(y+mx)$

$\Rightarrow$ C.F = $f_1(y-x) + f_2(y+2x) + x f_3(y+2x)$

P.I = $\frac{1}{f(D,D')} Q(x,y)$

$= \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$

Here $Q(x,y) = e^{x+2y}$ is of The form e^{ax+by}

so ~~find~~ put $D=a,\ D'=b$ in $f(D,D')$ we have

ie put $D=1,\ D'=2$ in

$f(D,D') = D^3 - 3D^2D' + 4D'^3$

$\Rightarrow f(a,b) = f(1,2) = 1 - 3\times 2 + 4\times 8 = 27 \neq 0.$

$\therefore$ P.I = $\frac{1}{f(a,b)} \iiint Q(u)\,du\,du\,du$ where $u = x+2y$.

$= \frac{1}{27} \iiint e^u\,du\,du\,du$

order of P.D.E is 3
so n = 3
Integrate 3 times

$$P.I = \frac{1}{27} e^{u} = \frac{1}{27} e^{x+2y}$$

∴ Complete sol'n $z = C.F + P.I$

$$\Rightarrow z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{1}{27} e^{x+2y}$$

② Solve $(D^3 - 4D^2D' + 4DD'^2)z = 6\sin(3x+2y)$

Sol:- Here $f(D,D') = D^3 - 4D^2D' + 4DD'^2$, $Q(x,y) = 6\sin(3x+2y)$

A.E is $f(m,1) = 0$

$\Rightarrow m^3 - 4m^2 + 4m = 0$

$\Rightarrow m(m^2 - 4m + 4) = 0$

$\Rightarrow m(m-2)^2 = 0$

$\Rightarrow m = 0, 2, 2$

i.e $m_1 = 0$, $m_2 = m_3 = 2 = m$

Here one root is different root and remaining two roots are equal roots

$C.F = f_1(y+m_1x) + f_2(y+mx) + x f_3(y+mx)$

$C.F = f_1(y) + f_2(y+2x) + x f_3(y+2x)$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^3 - 4D^2D' + 4DD'^2} \cdot 6\sin(3x+2y)$$ (Compare $\sin(ax+by)$)

Here $a=3$, $b=2$, $n = \text{order} = 3$

put $D = a$, $D' = b$

$f(D,D') = f(a,b) = 3^3 - 4\times3^2\times2 + 4\times3\times2^2 = 3 \neq 0$.

$$P.I = 6\times\frac{1}{f(a,b)} \iiint \sin u \, du\, du\, du \quad , \quad u = ax+by$$

$$= \frac{6}{3} \iiint \sin u \, du\, du\, du$$

$$= 2 \iint (-\cos u)\, du\, du$$

$$= 2 \int (-\sin u)\, du$$

$$P.I = 2\cos u = 2\cos(3x+2y)$$

∴ Complete sol'n $z = C.F + P.I$

$$z = \underline{f_1(y) + f_2(y+2x)} + \underline{x f_3(y+2x)} + 2\cos(3x+2y)$$

③ Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x$

Sol:- Symbolic form of the given Homo P.D.E is

$$(D^2 - 2DD' + D'^2)\, z = \sin(x+0y)$$

Here $f(D,D') = D^2 - 2DD' + D'^2$, $Q(x,y) = \sin(x+0y)$

A.E is $f(m,1)=0 \Rightarrow m^2-2m+1=0$

$\Rightarrow (m-1)^2=0$

$\Rightarrow m=1,1$ are two equal roots

$C.F = f_1(y+mx) + x f_2(y+mx)$

$\Rightarrow C.F = f_1(y+x) + x f_2(y+x)$.

$P.I = \frac{1}{f(D,D')} Q(x,y)$

$= \frac{1}{D^2-2DD'+D'^2}\sin(x+0y)$ (It is of the form $\sin(ax+by)$)

put $D=a=1$, $D'=b=0$ in $f(D,D')$ we have, $n=$ order $=2$

$f(D,D') = f(a,b) = f(1,0) = 1-0+0 = 1 \neq 0$

$P.I = \frac{1}{f(a,b)} \iint \sin u \, du\, du$, $u = x+0y$

$P.I = \frac{1}{1} \iint \sin u \, du\, du$

$P.I = -\sin u = -\sin x$.

G.S is $z = C.F + P.I = f_1(y+x) + x f_2(y+x) - \sin x$.

④ Solve $r + s - 2t = \sqrt{2x+y}$

Sol:- Given Homo P.D.E is $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = \sqrt{2x+y}$

Symbolic form of given P.D.E is $(D^2+DD'-2D'^2)z = \sqrt{2x+y}$

Here $f(D,D') = D^2+DD'-2D'^2$

$Q(x,y) = \sqrt{2x+y}$

⑪ A·E is $f(m,1)=0$

$\Rightarrow m^2+m-2=0$

$\Rightarrow (m+2)(m-1)=0$

$\Rightarrow m_1=-2, m_2=1$ are different roots

$C\cdot F = f_1(y+m_1x) + f_2(y+m_2x)$

$\Rightarrow C\cdot F = f_1(y-2x) + f_2(y+x)$.

$$P\cdot I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2+DD'-2D'^2} \sqrt{2x+y}$$

Here $a=2, b=1, n=\frac{1}{2}$, put $D=a$, $D'=b$ in $f(D,D')$

Now $f(D,D')=f(a,b)=f(2,1)=4+2-2=4 \neq 0$.

$$P\cdot I = \frac{1}{f(a,b)} \iint \sqrt{u}\, du\, du, \quad u=2x+y$$

$$= \frac{1}{4} \iint u^{1/2}\, du\, du$$

$$= \frac{1}{4} \int \frac{u^{3/2}}{3/2}\, du$$

$$= \frac{1}{4} \times \frac{2}{3} \times \frac{u^{5/2}}{5/2}$$

$$P\cdot I = \frac{1}{4} \times \frac{2}{3} \times \frac{2}{5}\, u^{5/2} \quad (\text{replace } u)$$

$$\Rightarrow P\cdot I = \frac{1}{15}(2x+y)^{5/2}$$

∴ Complete sol'n $z = C\cdot F + P\cdot I$

$$z = \underline{f_1(y-2x) + f_2(y+x) + \frac{1}{15}(2x+y)^{5/2}}$$

⑤ Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$

Sol:- Symbolic form of given Homo P.D.E is

$(D^2 - DD')z = \sin x \cos 2y$

$f(D,D') = D^2 - DD'$ $\quad Q(x,y) = \sin x \cos 2y$

$$= \frac{1}{2}(2 \sin x \cos 2y)$$

$$= \frac{1}{2}\left[\sin(x+2y) + \sin(x-2y)\right]$$

⑫

A·E is $f(m,1)=0$

$\Rightarrow m^2-m=0$

$\Rightarrow m(m-1)=0$

$\Rightarrow m_1=0,\ m_2=1$ are different roots

$C\cdot F = f_1(y+m_1x)+f_2(y+m_2x)$

$C\cdot F = f_1(y)+f_2(y+x)$.

$P\cdot I = \frac{1}{f(D,D')}\,\phi(x,y)$

$= \frac{1}{f(D,D')}\ \frac{1}{2}\left[\sin(x+2y)+\sin(x-2y)\right]$

$= \frac{1}{2}\left[\frac{1}{f(D,D')}\sin(x+2y)+\frac{1}{f(D,D')}\sin(x-2y)\right] \leftarrow ①$

Now $\frac{1}{f(D,D')}\sin(x+2y) = \frac{1}{D^2-DD'}\sin(x+2y)$

Here $a=1,\ b=2,\ n=2$ (order)

put $D=a=1,\ D'=b=2$ in $f(D,D')$ we have

$f(D,D')=f(a,b)=f(1,2)=1-1\times2=-1\neq0$

$\frac{1}{f(D,D')}\sin(x+2y)=\frac{1}{-1}\iint \sin u\,du\,du,\ u=x+2y$

$= \sin u = \sin(x+2y)$

lly $\frac{1}{f(D,D')}\sin(x-2y) = \frac{1}{D^2-DD'}\sin(x-2y)$

Here $a=1,\ b=-2,\ n=2$

$f(D,D')=f(a,b)=f(1,-2)=1^2-1(-2)=3\neq0$

$\frac{1}{f(D,D')}\sin(x-2y)=\frac{1}{f(a,b)}\iint \sin u\,du\,dy,\ u=x-2y$

$=\frac{1}{3}\iint \sin u\,du\,dy = -\frac{1}{3}\sin u$

Sub these in ①, we have $= \frac{1}{3}\sin(x-2y)$

$P\cdot I = \frac{1}{2}\left[\sin(x+2y)-\frac{1}{3}\sin(x-2y)\right]$

G·S is $z = C\cdot F + P\cdot I = f_1(y)+f_2(y+x)+\frac{1}{2}\sin(x+2y)-\frac{1}{6}\sin(x-2y)$

⑬ H.W solve $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + e^{2x+y}$

A) $z = f_1(y-x) + f_2(y+3x) + f_3(y-2x) - \frac{1}{75}\cos(x+2y) - \frac{1}{12}e^{2x+y}$

⑥ solve $(D^2 - 2DD' + D'^2)z = e^{x+y}$

Sol: $f(D,D') = D^2 - 2DD' + D'^2$, $Q(x,y) = e^{x+y}$

A.E is $f(m,1) = 0$

$\Rightarrow m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$ are equal roots

$\therefore C.F = f_1(y+mx) + x f_2(y+mx)$

$\Rightarrow C.F = f_1(y+x) + x f_2(y+x)$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2 - 2DD' + D'^2} e^{x+y}$$ (Compare with e^{ax+by})

Here $a=1$, $b=1$, $n=2$

put $D=a$, $D'=b$ in $f(D,D')$ we have.

$f(D,D') = f(a,b) = f(1,1) = 1^2 - 2\times1\times1 + 1^2 = 0$

It is a case of failure,

in this case $P.I = x.\frac{1}{\frac{\partial}{\partial D}[D^2 - 2DD' + D'^2]} e^{x+y}$

$$= x.\frac{1}{2D - 2D'} e^{x+y}$$

Here $a=1$, $b=1$, $n=1$

put $D=a$, $D'=b$ in $2D-2D'$ we have.

$2D - 2D' = 2\times1 - 2\times1 = 0$

again it is case of failure.

$$P.I = x^2.\frac{1}{\frac{\partial}{\partial D}(2D - 2D')} \cdot e^{x+y}$$

$= x^2.\frac{1}{2} \cdot e^{x+y}$ (partial derivative w.r.to D, D' is constant)

$$P.I = \frac{x^2}{2} e^{x+y}$$

$\therefore$ G.S is $z = C.F + P.I.$

⑭

⑦ solve $r+s-6t = \cos(2x+y)$

Sol:- Given $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \cos(2x+y)$

symbolic form is $(D^2+DD'-6D'^2)z = \cos(2x+y)$

Here $f(D,D') = D^2+DD'-6D'^2$, $Q(x,y) = \cos(2x+y)$

A.E is $f(m,1)=0 \Rightarrow m^2+m-6=0 \Rightarrow (m+3)(m-2)=0$

$\Rightarrow m_1 = -3,\ m_2 = 2$ are different roots

$C.F = f_1(y+m_1x) + f_2(y+m_2x)$

$\Rightarrow C.F = f_1(y-3x) + f_2(y+2x)$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2+DD'-6D'^2} \cos(2x+y)$$

Here $a=2,\ b=1,\ n=2$

put $D=a=2,\ D'=b=1$ in $f(D,D')$ we have

$f(a,b) = f(2,1) = 2^2+2\times1-6\times1^2=0$

It is a case of failure, in this case

$$P.I = x \cdot \frac{1}{\frac{\partial}{\partial D}[D^2+DD'-6D'^2]} \cos(2x+y)$$

$$= x \cdot \frac{1}{2D+D'} \cos(2x+y)$$

put $D=a=2,\ D'=b=1,\ n=1$ (order of $2D+D'$ is 1)

in the above, we have $f(2,1)=5\neq0$

$P.I = x \cdot \frac{1}{u+1} \int \cos u\, du$, $u=2x+y$

$= \frac{x}{5} \sin u$

$\Rightarrow P.I = \frac{x}{5} \sin(2x+y)$

G.S is $z = C.F + P.I$.

(5)

8. Solve $(D^2+5DD'+6D'^2)z = \frac{1}{y-2x}$

Sol:- Here $f(D,D') = D^2+5DD'+6D'^2$, $Q(x,y) = \frac{1}{y-2x}$

A.E is $f(m,1)=0$

$\Rightarrow m^2+5m+6=0 \Rightarrow (m+2)(m+3)=0$

$\Rightarrow m_1=-2,\ m_2=-3$ are different roots

$C.F = f_1(y+m_1x) + f_2(y+m_2x)$

$\Rightarrow C.F = f_1(y-2x) + f_2(y-3x)$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2+5DD'+6D'^2}\left(\frac{1}{y-2x}\right) \quad \left(\text{Compare with } (ax+by)^{-1}\right)$$

Here $a=-2, b=1, n=2$

put $D=a$, $D'=b$ in $f(D,D')$ we have

$f(D,D') = f(a,b) = f(-2,1) = 4-10+6 = 0$.

It is a case of failure, in this case

$$P.I = x.\frac{1}{\frac{\partial}{\partial D}[D^2+5DD'+6D'^2]}(y-2x)^{-1}$$

$$= x.\frac{1}{2D+5D'}(y-2x)^{-1}$$

Here $a=-2, b=1, n=1$

put $D=a$, $D'=b$ in $2D+5D'$ we have

$2D+5D' = 2(-2)+5(1) = 1 \neq 0$

$\therefore P.I = x.\frac{1}{(1)}\int u^{-1}\,du,\quad u=y-2x$

$= x\int \frac{1}{u}\,du$

$P.I = x\log u = x\log(y-2x)$

$\therefore$ G.S is $z = C.F + P.I$

$z = f_1(y-2x) + f_2(y-3x) + x\log(y-2x)$.

⑯

9. Solve $4r-4s+t = 16\log(x+2y)$

Sol:- Given P.D.E $4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial^2 z}{\partial y^2} = 16\log(x+2y)$

Symbolic form of given P.D.E is

$$(4D^2-4DD'+D'^2)z = 16\log(x+2y)$$

Here $f(D,D') = 4D^2-4DD'+D'^2$, $Q(x,y) = 16\log(x+2y)$

A.E is $f(m,1)=0 \Rightarrow 4m^2-4m+1=0$

$\Rightarrow (2m-1)^2=0 \Rightarrow m=\frac{1}{2},\frac{1}{2}$ are 2 equal roots

$C.F = f_1(y+mx) + xf_2(y+mx)$

$\Rightarrow C.F = f_1(y+\frac{1}{2}x) + x\, f_2(y+\frac{1}{2}x)$ (or)

$C.F = f_1(2y+x) + x\, f_2(2y+x)$

$P.I = \frac{1}{f(D,D')} Q(x,y)$

$= \frac{1}{4D^2-4DD'+D'^2} 16\log(x+2y)$ (Compare with $\log(ax+by)$)

Here $a=1$, $b=2$, $n=2$

Sub $D=a=1$, $D'=b=2$ in $f(D,D')$, we have.

$f(1,2) = 4\times1^2-4\times1\times2+2^2 = 8-8=0$

It is a case of failure, in this case

$$P.I = 16x.\frac{1}{\frac{\partial}{\partial D}[4D^2-4DD'+D'^2]}\log(x+2y)$$

$$= 16x.\frac{1}{8D-4D'}\log(x+2y)$$

put $D=a=1$, $D'=b=2$ in $8D-4D'$, $n=1$

we have $8D-4D' = 8\times1-4\times2=0$, Case of failure

$$P.I = 16x^2.\frac{1}{\frac{\partial}{\partial D}(8D-4D')}\log(x+2y)$$

$$= 16x^2.\frac{1}{8}\log(x+2y)$$

$P.I = 2x^2\log(x+2y)$

$\therefore z = C.F + P.I$

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(17)

⑩ Solve $(D-2D')(D-D')^{v} z = e^{x+y}$

Sol:- Here $f(D,D') = (D-2D')(D-D')^{v}$, $Q(x,y) = e^{x+y}$

A.E is $f(m,1)=0$

$\Rightarrow (m-2)(m-1)^{v}=0 \Rightarrow m_1=2, \; m_2=m_3=1,=m$

Here one root is different root and remaining two roots are equal roots

$\therefore C.F = f_1(y+m_1x) + f_2(y+mx) + x f_3(y+mx)$

$\Rightarrow C.F = f_1(y+2x) + f_2(y+x) + x f_3(y+x)$.

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{(D-D')^{v}(D-2D')} e^{x+y}$$

$$= \frac{1}{(D-D')^{v}} \left\{ \frac{1}{D-2D'} e^{x+y} \right\}$$

2nax+by

$$= \frac{1}{(D-D')^{v}} \left\{ \frac{1}{1-2} \int e^{u} du \right\} \quad (\because a=1, b=1, n=1)$$

$$= \frac{1}{(D-D')^{v}} \left\{ -e^{u} \right\}$$

$$= \frac{-1}{(D-D')^{v}} e^{x+y}$$

$$= \frac{-1}{0} e^{x+y}$$, Case of failure put $D=1, D'=1$

$$= -x \cdot \frac{1}{\frac{\partial}{\partial D}[D-D']^{v}} e^{x+y}$$

$$= -x \cdot \frac{1}{2(D-D')} e^{x+y}$$

$$= \frac{-x}{2} \cdot \frac{1}{\underset{\to 0}{(1-1)}} e^{x+y}$$ again case of failure

$$P.I = \frac{-x}{2} \left\{ x \cdot \frac{1}{\frac{\partial}{\partial D}(D-D')} e^{x+y} \right\} = \frac{-x^{v}}{2} e^{x+y}$$

$\therefore z = C.F + P.I = f_1(y+2x) + f_2(y+x) + x f_3(y+x) - \frac{x^{v}}{2} e^{x+y}$

P.I of $f(D,D')z = Q(x,y)$ when $Q(x,y) = x^m y^n$

when $Q(x,y) = x^m y^n$ then

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{f(D,D')} x^m y^n$$

$$P.I = [f(D,D')]^{-1} x^m y^n$$

If $m<n$, expand $[f(D,D')]^{-1}$ in powers of $\frac{D}{D'}$

ie take common D' from $f(D,D')$

If $n<m$, expand $[f(D,D')]^{-1}$ in powers of $\frac{D'}{D}$

ie take common D from $f(D,D')$

Also we have $\frac{1}{D} Q(x,y) = \int\limits_{y \text{ constant}} Q(x,y)\, dx$

$$\frac{1}{D'} Q(x,y) = \int\limits_{x \text{ constant}} Q(x,y)\, dy$$

Note ①^{2n} $Q(x,y) = x^m y^n$, (i) If x power is highest then take common D from $f(D,D')$

(ii). If y power is highest then take common D' from $f(D,D')$.

Problems ① solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2 y$

Sol:- symbolic form of the given eqn is put $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

$$(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y.$$

Here $f(D,D') = D^3 - 2D^2D'$, $Q(x,y) = 2e^{2x} + 3x^2y$

A.E is $f(m,1) = 0$

$\Rightarrow m^3 - 2m^2 = 0$

$\Rightarrow m^2(m-2) = 0$

$\Rightarrow m_1 = 2,\ m_2 = m_3 = 0$

Here one root is different and remaining two roots are equal.

$$CF = f_1(y+m_1x) + f_2(y+m_2x) + x f_3(y+m_3x).$$

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$C\cdot F = f_1(y+2x) + f_2(y) + x f_3(y).$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{f(D,D')}\left(2e^{2x} + 3x^2y\right)$$

$$= \left[2\cdot\frac{1}{f(D,D')} e^{2x} + 3\frac{1}{f(D,D')} x^2y\right]$$

$$P.I = 2P_1 + 3P_2 \quad \text{—①}$$

Now $2P_1 = 2\cdot\frac{1}{f(D,D')} e^{2x}$

$$= 2\cdot\frac{1}{D^3 - 2D^2D'} e^{2x+0y}$$ (Compare with e^{ax+by})

Here $a=2$, $b=0$, $n=3$ (order)

put $D=a=2$, $D'=b=0$ in $f(D,D')$, we have

$f(D,D') = f(2,0) = 2^3 - 2\times2^2\times0 = 8 \neq 0$

$\therefore 2P_1 = 2\cdot\frac{1}{8}\iiint e^u\, du\,du\,du$, $u = 2x+0y$

$$= \frac{1}{4}e^u \Rightarrow 2P_1 = \frac{1}{4}e^{2x}.$$

$$2P_2 = 3\frac{1}{f(D,D')} x^2y$$

$$= 3\cdot\frac{1}{D^3 - 2D^2D'} x^2y$$ (Compare with $x^m y^n$)

$m=2$, $n=1$, $m>n$ (ie x power is high).

So take common highest power of D from $f(D,D')$, we have

$$= 3\cdot\frac{1}{D^3\left[1-\frac{2D'}{D}\right]} x^2y.$$

$$= 3\cdot\frac{1}{D^3}\left[1-\frac{2D'}{D}\right]^{-1} x^2y.$$

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$$zP_2 = \frac{3}{D^3}\left[1+\frac{2D'}{D}+\left(\frac{2D'}{D}\right)^2+\cdots\right]x^2y$$

$(\because (1-x)^{-1} = 1+x^2+x^2+\cdots$

$$= \frac{3}{D^3}\left[1+\frac{2D'}{D}\right]x^2y$$

(neglect ~~higher power~~ from D'^2 and higher power since y power is one so consider upto D' power is one)

$$= \frac{3}{D^3}\left[x^2y+\frac{2D'}{D}(x^2y)\right]$$

$$= \frac{3}{D^3}\left[x^2y+\frac{2}{D}x^2\,D'(y)\right]$$

$D'y=1,\ D''y = D'''y = \cdots =$

→ w.r.to D', D is constant

$$= \frac{3}{D^3}\left[x^2y+\frac{2}{D}x^2\right]$$

$$= \frac{3}{D^3}\left[x^2y+2\cdot\frac{x^3}{3}\right]$$

$\left[\frac{1}{D} \to \int dx.\right]$

$$= 3y\iiint x^2\,dx\,dx\,dx + \cancel{6}\cdot 2\iiint x^3\,dx\,dx\,dx$$

$$= 3y\cdot\left(\frac{x^5}{60}\right) + 2\cdot\left(\frac{x^6}{120}\right)$$

$$zP_2 = \frac{yx^5}{20} + \frac{x^6}{60}$$

Sub zP_1, zP_2 in ①, we have

$$P.I = \frac{1}{4}e^{2x} + y\frac{x^5}{20} + \frac{x^6}{60}$$

∴ G.S is $z = C.F + P.I$

$$z = f_1(y+2x) + f_2(y) + x\,f_3(y) + \frac{1}{4}e^{2x} + \frac{yx^5}{20} + \frac{x^6}{60}.$$

Note

$\frac{1}{D^3} \to \iiint dx\,dx\,dx.$ lly $\frac{1}{D'^3} = \iiint dy\,dy\,dy$

$\frac{1}{D^2} \to \iint dx\,dx$ $\frac{1}{D'^2} = \iint dy\,dy.$

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② solve $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$.

Sol:- Here $f(D,D') = D^2 - 6DD' + 9D'^2$, $\phi(x,y) = 12x^2 + 36xy$

A.E is $f(m,1) = 0$ (put $D = m, D' = 1$)

$\Rightarrow m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0$

$\Rightarrow m = 3, 3$ are two equal roots

$\therefore C.F = f_1(y+mx) + x f_2(y+mx)$.

$\Rightarrow C.F = f_1(y+3x) + x f_2(y+3x)$

$$P.I = \frac{1}{f(D,D')}\phi(x,y)$$

$$= \frac{1}{f(D,D')}\left[12x^2 + 36xy\right] = 12 \cdot \frac{1}{f(D,D')}x^2 + 36 \cdot \frac{1}{f(D,D')}xy$$

$= z_{P_1} + z_{P_2}$ — ①

$$z_{P_1} = 12 \cdot \frac{1}{f(D,D')}x^2$$

$$= 12 \cdot \frac{1}{D^2 - 6DD' + 9D'^2}(x+0y)^2$$ (compare with $(ax+by)$)

$a = 1, b = 0$, n = order of D.E = 2

put $D = a = 1$, $D' = b = 0$ in $f(D,D')$, we have

$f(D,D') = f(1,0) = 1 \neq 0$.

$$z_{P_1} = 12 \cdot \frac{1}{1}\iint u^2\,du\,du$$ $(u = x)$.

$$= 12 \cdot \frac{u^4}{12} \Rightarrow z_{P_1} = (x)^4 = \cdot x^4$$

$$z_{P_2} = 36 \cdot \frac{1}{f(D,D')}xy.$$

$$= 36 \cdot \frac{1}{D^2 - 6DD' + 9D'^2}(xy)$$ Compare $(x^m y^n)$

$$= 36\frac{1}{(D-3D')^2}(xy)$$

Here $m = 1$, $n = 1$ so $m = n$.

Hence take Higher power of D or higher power of D' Common from $f(D,D')$

Here I take highest power of D Common from $f(D,D')$

㉒ $zP_2 = 36 \cdot \frac{1}{D^2\left[1-\frac{3D'}{D}\right]^2} xy$

$= \frac{36}{D^2}\left[1-\frac{3D'}{D}\right]^{-2}(xy)$

$= \frac{36}{D^2}\left[1+2\left(\frac{3D'}{D}\right)+\cdots\right](xy)$ $\left((1-x)^{-2}=1+2x+3x^2+\cdots\right)$

y power is one to Consider upto D' power is one

$= \frac{36}{D^2}\left[xy+\frac{6}{D}D'(xy)\right]$

$= \frac{36}{D^2}\left[xy+\frac{6}{D}x\right]$

$= 36y\,\frac{1}{D^2}(x) + 36\times 6\,\frac{1}{D^3}(x)$

$= 36y\iint x\,dx\,dx + 36\times 6\iiint x\,dx\,dx\,dx.$

$= 36y\left(\frac{x^3}{6}\right) + 36\times\cancel{6}\left(\frac{x^4}{\cancel{24}\,4}\right)$

$zP_2 = 6x^3y + 9x^4.$

sub zP_1, zP_2 in ①, we have

$P.I = x^4+6x^3y+9x^4 = 6x^3y+10x^4$

$\therefore$ G.S is $z = C.F + P.I$

$\Rightarrow z = f_1(y+3x)+x f_2(y+3x)+6x^3y+10x^4.$

⑬ $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x^2+y^2$

sol:- Symbolic form of the given D.E is $(D^2-D'^2)z = x^2+y^2$

Here $f(D,D') = D^2-D'^2$, $Q(x,y) = x^2+y^2$

A.E is $f(m,1)=0 \Rightarrow m^2-1=0$

$\Rightarrow m=\pm 1$ ie $m_1=-1$, $m_2=1$ are different roots

㉓ $C\cdot F = f_1(y+m_1x) + f_2(y+m_2x)$

$\Rightarrow C\cdot F = f_1(y-x) + f_2(y+x).$

$$P\cdot I = \frac{1}{f(D,D')}\,Q(x,y)$$

$$= \frac{1}{D^2-D'^2}\,(x^2+y^2)$$

$$= \frac{1}{D^2-D'^2}\,(x^2) + \frac{1}{D^2-D'^2}\,(y^2) = \partial P_1 + \partial P_2 \quad \text{—①}$$

Compare $(ax+by)$

$$\partial P_1 = \frac{1}{D^2-D'^2}\,(x^2) = \frac{1}{D^2-D'^2}\,(x+0y)^2$$

$a=1,\ b=0,\ n=2.$

put $D=a=1,\ D'=b=0$ in $f(D,D')$

$f(D,D') = f(1,0) = 1 \neq 0$

$$\partial P_1 = \frac{1}{1}\iint u^2\,du\,du \quad \text{where } u = x+0y$$

$$= \frac{u^4}{12} = \frac{x^4}{12}$$

$$\partial P_2 = \frac{1}{D^2-D'^2}\,(y^2) = \frac{1}{D^2-D'^2}\,(0x+y)^2$$

$a=0,\ b=1,\ n=2$ (order of D·E)

$f(D,D') = f(0,1) = 0^2-1^2 = -1 \neq 0$

$$\partial P_2 = \frac{1}{-1}\iint u^2\,du\,du, \quad \text{where } u = 0x+y$$

$$= -\frac{u^4}{12} = -\frac{y^4}{12}$$

$$\iint u^2\,du\,du = \int\left[\int u^2\,du\right]du = \int\left(\frac{u^3}{3}\right)du = \frac{u^4}{12}$$

sub $\partial P_1, \partial P_2$ in ①, we have

$$P\cdot I = \frac{x^4}{12} - \frac{y^4}{12}$$

$\therefore$ G.S is $z = C.F + P.I$

$$\Rightarrow z = f_1(y-x) + f_2(y+x) + \frac{1}{12}(x^4-y^4).$$

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(24) ④ $$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2+xy+y^2$$

Sol:- Symbolic form of the given DE is

$$(D^2+2DD'+D'^2)\, z = x^2+xy+y^2$$

Here $f(D,D') = D^2+2DD'+D'^2$, $Q(x,y) = x^2+xy+y^2$

A.E is $f(m,1) = 0 \Rightarrow m^2+2m+1=0 \Rightarrow (m+1)^2=0$

$\Rightarrow m = -1,-1$ are two equal roots

$C.F = f_1(y+mx) + x f_2(y+mx)$

$\Rightarrow C.F = f_1(y-x) + x f_2(y-x).$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{f(D,D')}(x^2+xy+y^2)$$

$$= \frac{1}{f(D,D')}x^2 + \frac{1}{f(D,D')}(xy) + \frac{1}{f(D,D')}(y^2)$$

$$= zP_1 + zP_2 + zP_3 \quad ④$$

$$zP_1 = \frac{1}{f(D,D')}x^2 = \frac{1}{(D+D')^2}x^2$$

$$= \frac{1}{(D+D')^2}(x+0y)^2 \quad \text{(Compare with } ax+by)$$

$a=1,\ b=0,\ n=2$

$f(D,D') = f(a,b) = f(1,0) = (1+0)^2 = 1 \neq 0.$

$$zP_1 = \frac{1}{1}\iint u^2\, du\, du, \quad \text{where } u = x+0y$$

$$= \frac{u^4}{12} = \frac{x^4}{12}$$

$$zP_3 = \frac{1}{f(D,D')}y^2 = \frac{1}{(D+D')^2}(0x+y)^2$$

$a=0,\ b=1,\ n=2$

$f(D,D') = f(a,b) = (0+1)^2 = 1 \neq 0.$

$$zP_3 = \frac{1}{1}\iint u^2\, du\, du, \quad u = 0x+y$$

$$= \frac{u^4}{12} = \frac{y^4}{12}$$

$$zP_2 = \frac{1}{f(D, D')}(xy)$$

$$= \frac{1}{(D+D')^2}(xy)$$ (Compare with $x^m y^n$, here $m = n = 1$ so take highest power of D or take highest power of D') common from $f(D, D')$)

$$= \frac{1}{D^2\left[1+\frac{D'}{D}\right]^2}(xy)$$

$$= \frac{1}{D^2}\left[1+\frac{D'}{D}\right]^{-2}(xy)$$

$$= \frac{1}{D^2}\left[1 - \frac{2D'}{D} + \cdots\right](xy)$$

$$= \frac{1}{D^2}\left[xy - 2\frac{D'}{D}(xy)\right]$$

$$= \frac{1}{D^2}\left[xy - \frac{2}{D}(x) D'(y)\right]$$

$$= y\frac{1}{D^2}(x) - \frac{2}{D^3}(x)$$

$$= y\iint x\,dx\,dx - 2\iiint x\,dx\,dx\,dx$$

$$zP_2 = y\frac{x^3}{6} - 2\left(\frac{x^4}{24}\right)$$

$$zP_2 = \frac{yx^3}{6} - \frac{x^4}{12}$$

sub zP_1, zP_2, zP_3 value in ①, we have

$$P.I = \cancel{\frac{x^4}{12}} + \frac{yx^3}{6} - \cancel{\frac{x^4}{12}} + \frac{y^4}{12}$$

$$P.I = \frac{yx^3}{6} + \frac{y^4}{12}$$

G.S is $z = C.F + P.I$

$$= f_1(y-x) + x f_2(y-x) + \frac{yx^3}{6} + \frac{y^4}{12}$$

H.W

① $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 12xy$ A) $f_1(y-x) + f_2(y-2x) + 2x^3y - \frac{3}{2}x^4$

② $(D^2 - 2DD')z = e^{2x} + x^3y$ A) $f_1(y) + f_2(y+2x) + \frac{1}{4}e^{2x} + \frac{x^5 y}{20} + \frac{x^6}{60}$.

⑤ $\frac{\partial^3 z}{\partial x^3} \cancel{+} - \frac{\partial^3 z}{\partial y^3} = x^3 y$ ✓

Sol: ~~Stan~~ symbolic form of given D.E is $(D^3 - D'^3) z = x^3 y$ ✓

Here $f(D,D') = D^3 - D'^3$, $Q(x,y) = x^3 y$ ✓

A.E is $f(m,1) = 0$

$\Rightarrow m^3 - 1 = 0$

$\Rightarrow \cancel{(m+1)}\ (m-1)(m^2+m+1) = 0 \Rightarrow (m-1) = 0,\ m^2+m+1 = 0$

$a=1, b=1, c=1$

$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\Rightarrow m_1 = 1, \quad m = \frac{-1 \pm \sqrt{1^2 - 4\times1\times1}}{2}$

$m = \frac{-1 \pm \sqrt{-3}}{2}$

$\Rightarrow m_2 = \frac{-1 \pm i\sqrt{3}}{2}$

$\therefore m_1 = 1,\ m_2 = \frac{-1+i\sqrt{3}}{2},\ m_3 = \frac{-1-i\sqrt{3}}{2}$ are different roots

Let us take $m_1 = 1$, $m_2 = \frac{-1+i\sqrt{3}}{2} = \omega$, $m_3 = \frac{-1-i\sqrt{3}}{2} = \omega^2$ ✓

Since w.k.t the given eqn $m^3 - 1 = 0 \Rightarrow m^3 = 1$ means Cube roots of unity which are $1, \omega, \omega^2$ ✓

$\therefore C.F = f_1(y+m_1x) + f_2(y+m_2x) + f_3(y+m_3x)$

$\Rightarrow C.F = f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x)$

$P.I = \frac{1}{f(D,D')} Q(x,y)$

$= \frac{1}{(D^3 - D'^3)} x^3 y^2$ ✓ $\quad (x^m y^n,\ m=3, n=2)$

$m > n$ so take common highest power of D from $f(D,D')$ we have

$= \frac{1}{D^3\left[1 - \frac{D'^3}{D^3}\right]} x^3 y^2$

$= \frac{1}{D^3}\left[1 - \frac{D'^3}{D^3}\right]^{-1} x^3 y^2$

$= \frac{1}{D^3}\left[1 + \cancel{\frac{D'^3}{D^3}} + \cancel{\left(\frac{D'^3}{D^3}\right)^2} + \cdots\right] x^3 y^2$ ✓

$P.I = \frac{1}{D3}(x^3y^2) = y^2 \iiint x^3\,dx\,dx\,dx.$

$P.I = \frac{y^2 \cdot x^6}{120}$ $\quad \int$ ② $\overset{x^3}{\frac{x^4}{4}}$ Ⅱ $\frac{x^5}{20}$ Ⅲ $\frac{x^6}{120}.$

$\therefore$ G.S is $z = C.F + P.I$

$\Rightarrow z = f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x) + \frac{y^2x^6}{120}.$

General method for finding P.I

If $Q(x,y)$ is of a form different from the forms already discussed, Then use.

$$\frac{1}{D-mD'}Q(x,y) = \int Q(x, c-mx)\,dx$$

ie replace y by $c-mx$, and after integration replace c by $y+mx$.

procedure factorise $f(D,D')$ into linear factors, Then

$$P.I = \frac{1}{f(D,D')}Q(x,y)$$

$$= \frac{1}{(D-m_1D')(D-m_2D')\cdots}Q(x,y)$$

$$= \frac{1}{D-m_1D'}\cdot\frac{1}{D-m_2D'}\cdots Q(x,y)$$

Repeated application of the above rule gives The P.I

problems ① Solve $(D^2-DD'-2D'^2)z = (y-1)e^x.$

Sol: $f(D,D') = D^2 - DD' - 2D'^2$, $Q(x,y) = (y-1)e^x$

A.E is $f(m,1) = 0 \Rightarrow m^2 - m - 2 = 0$

$\Rightarrow (m-2)(m+1) = 0$

$\Rightarrow m_1 = 2, m_2 = -1$ are different roots

$C.F = f_1(y+m_1x) + f_2(y+m_2x)$

$= f_1(y+2x) + f_2(y-x)$

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$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2 - DD' - 2D'^2} (y-1)e^x$$

$$= \frac{1}{(D-2D')(D+D')} (y-1)e^x$$

$$= \frac{1}{D-2D'} \cdot \frac{1}{D+D'} (y-1)e^x$$

(Comparing $D+D'$ with $D-mD'$, $m=-1$

replace $y = c - mx$

$y = c + x$

$\Rightarrow \boxed{c = y - x}$)

$$= \frac{1}{D-2D'} \int \underset{v}{e^x} \, \underset{u}{(c+x-1)} \, dx$$

$$= \frac{1}{D-2D'} \left\{ u \int v\,dx - \int \left(\frac{du}{dx} \int v\,dx \right) dx \right\}$$

$$= \frac{1}{D-2D'} \left\{ (c+x-1) \int e^x\,dx - \int \left[\frac{d}{dx}(c+x-1) \int e^x\,dx \right] dx \right\}$$

$$= \frac{1}{D-2D'} \left\{ (c+x-1)e^x - \int 1 \times e^x\,dx \right\}$$

$$= \frac{1}{D-2D'} \left\{ (c+x-1)e^x - e^x \right\}$$

replace $c = y - x$, we have

$$= \frac{1}{D-2D'} \left\{ (y - \cancel{x} + \cancel{x} - 1)e^x - e^x \right\}$$

$$= \frac{1}{D-2D'} \left[(y-1)e^x - e^x \right]$$

$$= \frac{1}{D-2D'} \left[(y-2)e^x \right]$$

Compare $D-2D'$ with $D-mD'$, here $m=2$

replace y by $c - mx = c - 2x$.

after integration replace $\underline{c = y + 2x}$.

$$P.I = \int \underset{u}{(c-2x-2)}\,\underset{v}{e^{x}}\,dx$$

$$= u\int v\,dx - \int\left[\frac{dy}{dx}\int v\,dx\right]dx$$

$$= (c-2x-2)\int e^{x}dx - \int\left[\frac{d}{dx}(c-2x-2)\int e^{x}dx\right]dx.$$

$$= (c-2x-2)\,e^{x} - \int(-2)\,e^{x}\,dx$$

$$= (c-2x-2)\,e^{x} + 2e^{x}$$

$$P.I = (c-2x-2+2)\,e^{x}$$

$$= (c-2x)\,e^{x}$$

$$P.I = (y+2x-2x)\,e^{x} \quad (\because\ c = y+2x)$$

$$\Rightarrow P.I = y e^{x}$$

$$\therefore G.S\ z = C.F + P.I$$

$$z = \underline{f_1(y+2x) + f_2(y-x) + y e^{x}}$$

② solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x\partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = y\cos x.$

Sol: Symbolic form of the given eqn is

$$(D^2 + DD' - 6D'^2)\,z = y\cos x.$$

Here $f(D,D') = D^2 + DD' - 6D'^2$, $Q(x) = y\cos x$.

A.E is $f(m,1) = 0 \Rightarrow m^2+m-6 = 0$

$\Rightarrow (m+3)(m-2)=0$

$\Rightarrow m_1 = -3,\ m_2 = 2$ are diff roots

$$\therefore C.F = f_1(y+m_1x) + f_2(y+m_2x)$$

$$\Rightarrow C.F = f_1(y-3x) + f_2(y+2x).$$

$$P.I = \frac{1}{f(D,D')}\,Q(x,y)$$

$$= \frac{1}{D^2+DD'-6D'^2}\,(y\cos x)$$

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$$P.I = \frac{1}{(D+3D')(D-2D')} (y \cos x)$$

$$= \frac{1}{D+3D'} \left\{ \frac{1}{D-2D'} (y \cos x) \right\}$$

Compare $D-2D'$ with $D-mD'$.

Here $m=2$

Put $y = c-mx$.

$\Rightarrow y = c-2x$.

replace after integration

$\boxed{c = y+2x}$

$$= \frac{1}{D+3D'} \int (\underset{u}{c-2x}) \underset{v}{\cos x}\, dx.$$

$$= \frac{1}{D+3D'} \left\{ u \int v\,dx - \int \left(\frac{dy}{dx} \int v\,dx \right) dx \right\}$$

$$= \frac{1}{D+3D'} \left\{ (c-2x) \int \cos x\, dx - \int \left[\frac{d}{dx} (c-2x) \int \cos x\, dx \right] dx \right\}$$

$$= \frac{1}{D+3D'} \left\{ (c-2x) \sin x - \int (-2) \sin x\, dx \right\}$$

$$= \frac{1}{D+3D'} \left\{ (c-2x) \sin x - 2 \cos x \right\}$$

$$= \frac{1}{D+3D'} \left\{ (y+\cancel{2x}-\cancel{2x}) \sin x - 2 \cos x \right\}$$

$$= \frac{1}{D-(-3D')} (y \sin x - 2 \cos x)$$

Here $m=-3$,

Put $y = c-mx$

$\Rightarrow y = c+3x$

$\Rightarrow \boxed{c = y-3x}$

$$= \int \left[(c+3x) \sin x - 2 \cos x \right] dx$$

$$= \int (\underset{u}{c+3x}) \underset{v}{\sin x}\, dx - 2 \int \cos x\, dx.$$

$$= u \int v\,dx - \int \left(\frac{dy}{dx} \int v\,dx \right) dx - 2 \sin x.$$

$$= (c+3x) \int \sin x\, dx - \int \left[\frac{d}{dx} (c+3x) \int \sin x\, dx \right] dx - 2 \sin x$$

$$= -(c+3x) \cos x - \int 3 (-\cos x)\, dx - 2 \sin x$$

$$= -(c+3x) \cos x + 3 \sin x - 2 \sin x$$

$$= -[y - \cancel{3x} + \cancel{3x}] \cos x + \sin x$$

$$\underline{P.I = -y \cos x + \sin x}$$

$$z = C.F + P.I$$

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(31) ③ Solve $(D^2+DD'-6D'^2)z = x^2\sin(x+y)$

Sol: Here $f(D,D') = D^2+DD'-6D'^2$, $\phi(x,y) = x^2\sin(x+y)$

A·E is $f(m,1)=0 \Rightarrow m^2+m-6=0$

$\Rightarrow (m+3)(m-2)=0$

$\Rightarrow m_1=-3,\ m_2=2$ are diff roots

C·F = $f_1(y+m_1x) + f_2(y+m_2x)$.

$\Rightarrow$ C·F = $f_1(y-3x) + f_2(y+2x)$.

$$P\cdot I = \frac{1}{f(D,D')}\phi(x,y)$$

$$= \frac{1}{(D+3D')(D-2D')}x^2\sin(x+y)$$

$$= \frac{1}{D+3D'}\left\{\frac{1}{D-2D'}x^2\sin(x+y)\right\}$$

Here $m=2$, put $y = c-mx$

$\Rightarrow y = c-2x$

$\therefore \boxed{c = y+2x}$

$$= \frac{1}{D+3D'}\int x^2\sin(x+c-2x)\,dx.$$

$$= \frac{1}{D+3D'}\int \underset{u}{x^2}\ \underset{v}{\sin(c-x)}\,dx.$$

$$= \frac{1}{D+3D'}\left[uv_1 - u'v_2 + u''v_3\right]$$ (Bernoullis rule $\int uv\,dx$)

$u = x^2$ $\quad v = \sin(c-x)$

$u' = 2x$ $\quad v_1 = \int v\,dx = -\frac{\cos(c-x)}{-1} = \cos(c-x)$

$u'' = 2$ $\quad v_2 = \int v_1\,dx = \frac{\sin(c-x)}{-1} = -\sin(c-x)$

$v_3 = \int v_2\,dx = -\left[\frac{-\cos(c-x)}{-1}\right] = -\cos(c-x)$

$$P\cdot I = \frac{1}{D+3D'}\left\{x^2\cos(c-x) - 2x\left[-\sin(c-x)\right] + 2\left[-\cos(c-x)\right]\right\}$$

$$= \frac{1}{D+3D'}\left\{x^2\cos(c-x) + 2x\sin(c-x) - 2\cos(c-x)\right\}$$

$$= \frac{1}{D+3D'}\left\{x^2\cos(y+2x-x) + 2x\sin(y+2x-x) - 2\cos(y+2x-x)\right.$$

$(\because c = y+2x)$

(32) $P.I = \frac{1}{D+3D'}\left\{x^2\cos(y+x) + 2x\sin(y+x) - 2\cos(y+x)\right\}$

Compare $D+3D'$ with $D-mD'$ Here $m=-3$, Put $y = c - mx$

$\rightarrow y = c+3x$

$\rightarrow \boxed{c = y-3x}$

~~$P.I = \int x^2\cos(y-3x+x) + 2x\sin($~~

$$P.I = \int\left[x^2\cos(c+3x+x) + 2x\sin(c+3x+x) - 2\cos(c+3x+x)\right]dx$$

$$P.I = \int x^2\cos(c+4x)dx + 2\int x\sin(c+4x)dx - 2\int\cos(c+4x)dx \quad ①$$

$\int \underset{u}{x^2}\,\underset{v}{\cos(4x+c)}dx = uv_1 - u'v_2 + u''v_3$

$u = x^2$ $\quad v = \cos(4x+c)$

$v_1 = \int v\,dx = \frac{\sin(4x+c)}{4}$

$u' = 2x$

$u'' = 2$

$v_2 = \int v_1\,dx = -\frac{\cos(4x+c)}{4^2} = -\frac{\cos(4x+c)}{16}$

$v_3 = \int v_2\,dx = -\frac{\sin(4x+c)}{4^3} = -\frac{\sin(4x+c)}{64}$

$$\int x^2\cos(4x+c)dx = x^2\frac{\sin(4x+c)}{4} - 2x\left[-\frac{\cos(4x+c)}{16}\right] + 2\left[-\frac{\sin(4x+c)}{64}\right]$$

$$= \frac{x^2}{4}\sin(4x+y-3x) + \frac{x}{8}\cos(4x+y-3x) - \frac{1}{32}\sin(4x+y-3x)$$

$$\int x^2\cos(4x+c)dx = \frac{x^2}{4}\sin(x+y) + \frac{x}{8}\cos(x+y) - \frac{1}{32}\sin(x+y)$$

$$\int\cos(c+4x)dx = \frac{\sin(4x+c)}{4} = \frac{1}{4}\sin(4x+y-3x) = \frac{1}{4}\sin(x+y)$$

$\int \underset{u}{x}\,\underset{v}{\sin(4x+c)}dx = uv_1 - u'v_2$

$u = x$ $\quad v = \sin(4x+c)$

$u' = 1$ $\quad v_1 = \int v\,dx = -\frac{\cos(4x+c)}{4}$

$v_2 = \int v_1\,dx = -\frac{\sin(4x+c)}{16}$

(33) $\int x \sin(4x+c)\,dx = x\left[\frac{-\cos(4x+c)}{4}\right] - \left[\frac{-\sin(4x+c)}{16}\right]$

$= \frac{-x}{4}\cos(4x+y-3x) + \frac{1}{16}\sin(4x+y-3x)$

$\int x \sin(4x+c)\,dx. = \frac{-x}{4}\cos(x+y) + \frac{1}{16}\sin(x+y)$

sub all these values in ①, we have

$P.I = \frac{x^2}{4}\sin(x+y) + \frac{x}{8}\cos(x+y) - \frac{1}{32}\sin(x+y) +$
$2\left[\frac{-x}{4}\cos(x+y) + \frac{1}{16}\sin(x+y)\right] - 2\left[\frac{1}{4}\sin(x+y)\right]$

$= \frac{x^2}{4}\sin(x+y) + \frac{x}{8}\cos(x+y) - \frac{1}{32}\sin(x+y) - \frac{x}{2}\cos(x+y)$
$+ \frac{1}{8}\sin(x+y) - \frac{1}{2}\sin(x+y)$

$= \sin(x+y)\left[\frac{x^2}{4} - \frac{1}{32} + \frac{1}{8} - \frac{1}{2}\right] + \cos(x+y)\left[\frac{x}{8} - \frac{x}{2}\right]$

$= \sin(x+y)\left[\frac{x^2}{4} + \frac{-1+4-16}{32}\right] + x\cos(x+y)\left[\frac{1-4}{8}\right]$

$P.I = \left(\frac{x^2}{4} - \frac{13}{32}\right)\sin(x+y) - \frac{3x}{8}\cos(x+y)$

∴ G.S is $z = C.F + P.I$

$= f_1(y-3x) + f_2(y+2x) + \left(\frac{x^2}{4} - \frac{13}{32}\right)\sin(x+y)$
$- \frac{3x}{8}\cos(x+y).$

④ – solve ~~r-s-2t~~ = H.W problems

① $(D^2+DD'-6D'^2)z = y\sin x$ A) $f_1(y+2x) + f_2(y-3x) - y\sin x - \cos x$

② $(D-D')(D+2D')z = (y+1)e^x$
A) $f_1(y+x) + f_2(y-2x) + ye^x$.

③ $(D^2+2DD'+D'^2)z = 2\sin y - x\cos y$

Non- Homogeneous Linear partial differential eqn's

In the eqn $f(D,D')z = Q(x,y)$, if the polynomial $f(D,D')$ in D, D' is not homogeneous.(ie having different ~~powers~~ orders) Then it is called a non homogeneous linear partial differential eqn's

Its complete soln is $z = C.F + P.I$

Method for finding C.F. when $\phi(D,D')$ can be factorised into linear factors :-

Resolve $f(D,D')$ into linear factors of the form $(D - mD' - \alpha)$. The C.F corresponding to the factor $D - mD' - \alpha$ is $e^{\alpha x}\phi(y+mx)$.

if $f(D,D') = (D-m_1D'-\alpha_1)(D-m_2D'-\alpha_2)\cdots(D-m_nD'-\alpha_n)$

$$C.F = e^{\alpha_1 x} f_1(y+m_1x) + e^{\alpha_2 x} f_2(y+m_2x) + \cdots + e^{\alpha_n x} f_n(y+m_nx)$$

In case of repeated factors

If $f(D,D') = (D-mD'-\alpha)^2$ then

$$C.F = e^{\alpha x} f_1(y+mx) + x e^{\alpha x} f_2(y+mx)$$

If $f(D,D') = (D-mD'-\alpha)^3$ then

$$C.F = e^{\alpha x} f_1(y+mx) + x e^{\alpha x} f_2(y+mx) + x^2 e^{\alpha x} f_3(y+mx).$$

Methods for finding P.I

Let the given eqn be $f(D,D')z = Q(x,y)$.

Then $P.I = \frac{1}{f(D,D')} Q(x,y)$

Case(i) when $Q(x,y) = e^{ax+by}$ and $f(a,b) \neq 0$

$$P.I = \frac{1}{f(a,b)} e^{ax+by}$$

ie replace D by a & D' by b.

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㉟ **Note** If $f(a,b)=0$ Then it is a Case of failure

in this Case $P.I = x.\dfrac{1}{\frac{\partial}{\partial D}[f(D,D')]}e^{ax+by}$ (or) $y\dfrac{1}{\frac{\partial}{\partial D'}[f(D,D')]}e^{ax+by}$

Case (ii) when $Q(x,y) =$ $\sin(ax+by)$ or $\cos(ax+by)$

Then $P.I = \dfrac{1}{f(D,D')}\sin(ax+by)$

put $D^2 = -(a^2)$, $D'^2 = -(b^2)$, $DD' = -(ab)$ providing the denominator is non-zero

Note If the denominator is zero then

$$P.I = x.\frac{1}{\frac{\partial}{\partial D}[f(D,D')]}\sin(ax+by) \text{ or } y.\frac{1}{\frac{\partial}{\partial D'}[f(D,D')]}\sin(ax+by)$$

use the above conditions providing denominator is non zero, if the denominator is zero then

$$P.I = x^2.\frac{1}{\frac{\partial^2}{\partial D^2}[f(D,D')]}\sin(ax+by) \text{ or } y^2.\frac{1}{\frac{\partial^2}{\partial D'^2}[f(D,D')]}\sin(ax+by)$$

Case (iii) when $Q(x,y) = x^m y^n$ where m and n are positive integers. $P.I = \dfrac{1}{f(D,D')}x^m y^n = [f(D,D')]^{-1}x^m y^n$

which can be evaluated after expanding $[f(D,D')]^{-1}$ in ascending powers of $\frac{D'}{D}$ when $m>n$ or $\frac{D}{D'}$ when $m<n$

Case (iv) when $Q(x,y) = e^{ax+by}V$ where V is a function of x and y.

$$P.I = \frac{1}{f(D,D')}e^{ax+by}V = e^{ax+by}\frac{1}{f(D+a,D'+b)}V$$

which can be evaluated by using the previous cases.

㊱ problems

① solve $(D-D'-1)(D-D'-2)z = e^{2x-y}$

Here $f(D,D') = (D-D'-1)(D-D'-2)$, $Q(x,y) = e^{2x-y}$

Compare each factor of $f(D,D')$ with $D-mD'-\alpha$, we have

$m_1 = 1, \alpha_1 = 1$, $m_2 = 1, \alpha_2 = 2$

$C.F = e^{\alpha_1 x} f_1(y+m_1x) + e^{\alpha_2 x} f_2(y+m_2x)$

$= e^{x} f_1(y+x) + e^{2x} f_2(y+x)$

$P.I = \frac{1}{f(D,D')} Q(x,y)$

$= \frac{1}{(D-D'-1)(D-D'-2)} e^{2x-y}$

put $D=2$, $D'=-1$, we have

$= \frac{1}{(2+1-1)(2+1-2)} e^{2x-y} = \frac{1}{2} e^{2x-y}$

∴ Complete soln $z = C.F + P.I$

$\Rightarrow z = e^{x} f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{2} e^{2x-y}$.

② solve $(D^2-DD'+D'-1)z = \cos(x+2y)$

Clearly this is non Homogeneous D.E since

Here $f(D,D')$ have different order terms $\left(\begin{array}{lcccc} & D^2 & DD' & D' & 1 \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ \text{Order} & 2 & 2 & 1 & 0 \end{array}\right)$

Here $f(D,D') = D^2-DD'+D'-1$, $Q(x,y) = \cos(x+2y)$

$\Rightarrow f(D,D') = D^2-1-DD'+D'$

$= (D+1)(D-1) - D'(D-1)$

$= (D-1)(D+1-D')$

$= (D-0.D'-1)(D-D'+1)$ are product of two different linear factors compare with $D-mD'-\alpha$, we have

㊲

$$C\cdot F = e^{\alpha_1 x} f_1(y+m_1 x) + e^{\alpha_2 x} f_2(y+m_2 x)$$

Here $m=0,\ \alpha_1=1,\ \ m_2=1,\ \ \alpha_2=-1$

$$\boxed{C\cdot F = e^{x} f_1(y) + e^{-x} f_2(y+x).}$$

$$P\cdot I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2-DD'+D'-1}\cos(x+2y) \quad \left[\cos(ax+by)\ \text{Model ②}\right]$$

$a=1,\ b=2$

put $D^2 = -(a^2) = -1,\quad DD' = -(ab) = -2$

$$P\cdot I = \frac{1}{-1+2+D'-1}\cos(x+2y) \qquad \frac{1}{D'}\to\int dy$$

$$= \frac{1}{D'}\cos(x+2y) = \int \cos(x+2y)\,dy$$

$(x \text{ constant})$

$$P\cdot I = \frac{\sin(x+2y)}{2}$$

Gsis $z = C\cdot F + P\cdot I$

$$z = e^{x} f_1(y) + e^{-x} f_2(y+x) + \frac{1}{2}\sin(x+2y)$$

③ Solve $[D^2 - D'^2 - 3D + 3D']z = xy$

Sol:- Here $f(D,D') = D^2 - D'^2 - 3D + 3D'$, $Q(x,y) = xy$

Here $f(D,D')$ is a non homogeneous D.E in D, D'.

$$f(D,D') = (D+D')(D-D') - 3(D-D')$$

$= (D-D')[D+D'-3]$ are product of two different linear factors compare with $D-mD'-\alpha$

Here $m_1=1,\ \alpha_1=0\ ;\ m_2=-1,\ \alpha_2=3.$

$$C\cdot F = e^{\alpha_1 x} f_1(y+m_1 x) + e^{\alpha_2 x} f_2(y+m_2 x)$$

$$= e^{0x} f_1(y+x) + e^{3x} f_2(y-x)$$

$$\boxed{C\cdot F = f_1(y+x) + e^{3x} f_2(y-x)}$$

$$P\cdot I = \frac{1}{f(D,D')}\,\phi(x,y)$$

$$= \frac{1}{(D-D')(D+D'-3)}\,xy$$

$$= \frac{1}{D\left[1-\frac{D'}{D}\right]\left\{-3\left(1-\frac{D+D'}{3}\right)\right\}}\,xy$$

(Here $x^m y^n$
$m=1=n$, both are
same so express
$\frac{D'}{D}$ or $\frac{D}{D'}$)

$$= \frac{-1}{3D}\left(1-\frac{D'}{D}\right)^{-1}\left[1-\left(\frac{D+D'}{3}\right)\right]^{-1}xy$$

$$= \frac{-1}{3D}\left(1+\frac{D'}{D}+\cdots\right)\left[1+\frac{D+D'}{3}+\left(\frac{D+D'}{3}\right)^2+\cdots\right]xy$$

$$= \frac{-1}{3D}\left[1+\frac{D'}{D}+\cdots\right]\left[1+\frac{D+D'}{3}+\cancel{\frac{D^2}{9}}+\cancel{\frac{D'^2}{9}}+\frac{2DD'}{9}+\cdots\right]xy$$

$$= \frac{-1}{3D}\left[1+\frac{D+D'}{3}+\frac{2DD'}{3}+\frac{D'}{D}+\frac{D'}{D}\left(\frac{D+D'}{3}\right)+\frac{D'}{D}\left(\frac{2DD'}{3}\right)\right]xy$$

(∵ $D^2(x)=0$, $D'^2(y)=0$
so neglecting second and
higher order derivatives
of D, D')

$$= \frac{-1}{3D}\left[1+\frac{D}{3}+\frac{D'}{3}+\frac{2DD'}{9}+\frac{D'}{D}+\frac{D'}{3}+\cancel{\frac{D'^2}{3D}}+\cancel{\frac{2D'^2}{3}}\right]xy.$$

$$= \frac{-1}{3D}\left[1+\frac{D}{3}+\frac{2D'}{3}+\frac{D'}{D}+\frac{2DD'}{9}\right]xy$$

$$= \frac{-1}{3D}\left[xy+\frac{1}{3}y+\frac{2}{3}x+\frac{x^2}{2}+\frac{2}{9}\right]$$

$DD'(xy)=D(x)\,D'(y)$
$=1$

$$= \frac{-1}{3}\left[\frac{x^2}{2}y+\frac{1}{3}yx+\frac{2}{3}\cdot\frac{x^2}{2}+\frac{x^3}{6}+\frac{2}{9}x\right]$$

$\frac{D'}{D}(xy)=\frac{1}{D}(x)\,D'(y)$
$=\int x\,d\cancel{y}x\,1$
$=\frac{x^2}{2}$

$$P\cdot I = \frac{-1}{3}\left[\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{3}+\frac{x^3}{6}+\frac{2}{9}x\right]$$

Complete soln is $z = C\cdot F + P\cdot I$

$$\Rightarrow z = f_1(y+x)+e^{3x}f_2(y-x)-\frac{1}{3}\left(\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{3}+\frac{x^3}{6}+\frac{2}{9}x\right).$$

(39) ④ Solve $(D-3D'-2)^2 z = 2e^{2x}\tan(y+3x)$

Sol:- Here $f(D,D') = (D-3D'-2)^2$ are product of two repeated linear factors Compare with $(D-mD'-\alpha)^2$

Here $m=3,\ \alpha=2$.

$$C.F = e^{\alpha x} f_1(y+mx) + x e^{\alpha x} f_2(y+mx)$$

$$= e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x).$$

Given $Q(x,y) = 2e^{2x}\tan(y+3x)$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{(D-3D'-2)^2} 2e^{2x}\tan(y+3x)$$

put $D = D+2,\ D' = D'+0$ (Using Model ④ $e^{ax+by} Q(x,y)$)

$$= 2e^{2x} \frac{1}{(D+\cancel{2}-3D'-\cancel{2})^2} \tan(y+3x)$$

$$= 2e^{2x}\left[\frac{1}{(D-3D')^2}\tan(y+3x)\right]$$ (Compare with $\tan(ax+by)$)

Now this eqn becomes to Homogeneous D.E

Here $f_1(D,D') = (D-3D')^2$,

put $D = a = 3,\ D' = b = 1,\ n = \text{order} = 2$

$f_1(D,D') = f(3,1) = (3-3)^2 = 0$, Case of failure

$$P.I = 2e^{2x}\, x. \frac{1}{\frac{\partial}{\partial D}(D-3D')^2}\tan(y+3x)$$

$$= \cancel{2}\, xe^{2x} \frac{1}{\cancel{2}(D-3D')}\tan(y+3x)$$

put $D = a = 3,\ D' = b = 1,\ D-3D' = 3-3 = 0$

again Case of failure.

$$P.I = x\, e^{2x} \cdot x \cdot \frac{1}{\frac{\partial}{\partial D}(D-3D')} \tan(y+3x)$$

$$P.I = x^2 e^{2x} \cdot \tan(y+3y)$$

G.S is $z = C.F + P.I$

$$\Rightarrow z = e^{2x} f_1(y+3x) + x\, e^{2x} f_2(y+3x) + x^2 e^{2x} \tan(y+3x)$$

⑤ $(D^2 - DD' - 2D)z = \sin(3x+4y)$

Sol: $f(D,D') = D^2 - DD' - 2D$, $Q(x,y) = \sin(3x+4y)$

~~$= D^2 - 2DD' + DD' - 2D = D^2 + DD' - 2DD' - 2D$~~

~~$= D^2 - 2D - 2DD' + DD' = D(D+D') - 2($~~

~~$= D(D-2)$~~ -

$$f(D,D') = D^2 - DD' - 2D$$
$$= D[D - D' - 2]$$
$$= (D - 0.D' - 0)(D - D' - 2)$$ Compare with $D - mD' - \alpha$

Here $m_1 = 0,\ \alpha_1 = 0,\ \ m_2 = 1,\ \alpha_2 = 2$

$$C.F = e^{\alpha_1 x} f_1(y + m_1 x) + e^{\alpha_2 x} f_2(y + m_2 x)$$
$$= e^{0x} f_1(y + 0\times x) + e^{2x} f_2(y + x)$$

$$C.F = f_1(y) + e^{2x} f_2(y+x).$$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2 - DD' - 2D} \sin(3x+4y)$$ ($\sin(ax+by)$, $a=3, b=4$)

put $D^2 = -(3^2) = -9$, $DD' = -(ab) = -12$

$$P.I = \frac{1}{-9+12-2D} \sin(3x+4y)$$

$$= \frac{1}{3-2D} \frac{(3+2D)}{(3+2D)} \sin(3x+4y)$$

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(Q1)

$$P.I = \frac{(3+2D)}{9-4D^2} \sin(3x+4y)$$

$$\text{put } D^2 = -(a^2) = -(3^2) = -9$$

$$= \frac{(3+2D)}{9-4(-9)} \sin(3x+4y)$$

$$\Rightarrow P.I = \frac{3\sin(3x+4y) + 2D[\sin(3x+4y)]}{45}$$

$$= \frac{3\sin(3x+4y) + 6\,\cancel{\sin}\cos(3x+4y)}{45}$$

$$P.I = \frac{1}{15}[\sin(3x+4y) + 2\cos(3x+4y)] \quad \left[\begin{aligned} &D(\sin(3x+4y)) \\ &= \frac{\partial}{\partial x}\sin(3x+4y) \\ &= 3\cos(3x+4y) \end{aligned}\right]$$

G.S is $z = C.F + P.I$

$$\Rightarrow z = f_1(y) + e^{2x} f_2(y+x) + \frac{1}{15}[\sin(3x+4y) + 2\cos(3x+4y)].$$

(6) $r - s + p = 1$

Sol:- W.K.T $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $p = \frac{\partial z}{\partial x}$.

Symbolic form of the given D.E is

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = 1$$

$$\Rightarrow (D^2 - DD' + D)z = 1$$

Here $f(D,D') = D^2 - DD' + D$, $Q(x,y) = 1 = e^{0x+0y}$

$$\Rightarrow f(D,D') = D[D - D' + 1]$$

$= (D - 0D' + 0)(D - D' + 1)$ are two different linear factors Compare with $(D - mD' - \alpha)$, we have

$m_1 = 0, \alpha_1 = 0$; $m_2 = 1, \alpha_2 = -1$

$$C.F = e^{\alpha_1 x} f_1(y + m_1 x) + e^{\alpha_2 x} f_2(y + m_2 x)$$

$$\Rightarrow C.F = e^{0x} f_1(y) + e^{-x} f_2(y+x)$$

$$C.F = f_1(y) + e^{-x} f_2(y+x).$$

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(42)

$$P \cdot I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2 - DD' + D} e^{0x+0y} \quad \left(e^{ax+by} \text{ Model } ① \right)$$

put $D=0, D'=0$, we have

$$P \cdot I = \frac{1}{0} e^{0x+0y}$$, it is case of failure

$$= x \cdot \frac{1}{\frac{\partial}{\partial D}(D^2 - DD' + D)} e^{0x+0y}$$

$$= x \cdot \frac{1}{(2D - D' + 1)} e^{0x+0y}$$

put $D=0, D'=0$, we have

$$P \cdot I = x \cdot e^{0x+0y} = x.$$

$$G \cdot S \ z = C \cdot F + P \cdot I$$

$$= f_1(y) + e^{-x} f_2(y+x) + x.$$

---

⑦ $(D+D'-1)(D+2D'-3) z = 4 + 3x + 6y$

Sol:- $f(D,D') = (D+D'-1)(D+2D'-3)$, $Q(x,y) = 4 + 3x + 6y$

$m_1 = -1, \alpha_1 = 1$ $\quad m_2 = -2, \alpha_2 = 3$ different

$$C \cdot F = e^{\alpha_1 x} f_1(y + m_1 x) + e^{\alpha_2 x} f_2(y + m_2 x)$$

$$C \cdot F = e^{x} f_1(y - x) + e^{3x} f_2(y - 2x)$$

(iii) Now $\frac{1}{f(D,D')} e^{0x+0y} = \frac{1}{(D+D'-1)(D+2D'-3)} e^{0x+0y}$

put $D=0$, $D'=0$, we have,

$= \frac{1}{3} e^{0x+0y} = \frac{1}{3}$.

$$\frac{1}{f(D,D')} x = \frac{1}{(D+D'-1)(D+2D'-3)} x$$

$$= \frac{1}{(-1)\left[1-(D+D')\right](-3)\left[1-\left(\frac{D+2D'}{3}\right)\right]} x$$

$$= \frac{1}{3} \left[1-(D+D')\right]^{-1} \left[1-\left(\frac{D+2D'}{3}\right)\right]^{-1} x.$$

$$= \frac{1}{3} \left[1+(D+D')+(D+D')^2+\cdots\right] \left[1+\frac{1}{3}(D+2D')+\frac{1}{9}(D+2D')^2+\cdots\right] x$$

$$= \frac{1}{3} \left[1+D+\cancel{D'}\right]\left[1+\frac{1}{3}D+\cancel{\frac{2}{3}D'}\right] x.$$

$$= \frac{1}{3} \left[1+D\right]\left[1+\frac{1}{3}D\right] x \qquad \left(\because D'(x)=0 \text{ i.e. } \frac{\partial}{\partial y}(x)=0\right)$$

$$= \frac{1}{3} \left[1+\frac{1}{3}D+D+\cancel{\frac{1}{3}D^2}\right] y \qquad \left(D^2(x)=0\right)$$

$$= \frac{1}{3}\left(1+\frac{4}{3}D\right)x = \underline{\frac{1}{3}\left(x+\frac{4}{3}\right)}$$

$$\frac{1}{f(D,D')} y = \frac{1}{(D+D'-1)(D+2D'-3)} (y)$$

$$= \frac{1}{(-1)\left[1-(D+D')\right](-3)\left[1-\left(\frac{D+2D'}{3}\right)\right]} (y)$$

$$= \frac{1}{3} \left[1-(D+D')\right]^{-1} \left[1-\left(\frac{D+2D'}{3}\right)\right]^{-1} (y)$$

$$= \frac{1}{3} \left[1+\cancel{D}+D'\right]\left[1+\cancel{\frac{D}{3}}+\frac{2}{3}D'\right] y \qquad \left[D(y)=0.\right]$$

$$= \frac{1}{3}(1+D')\left(1+\frac{2}{3}D'\right) y$$

$$= \frac{1}{3}\left(1+\frac{2}{3}D'+D'+\frac{2}{3}\cancel{D'^2}\right) y \qquad D'^2(y)=0$$

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⑭ $\frac{1}{f(D,D')}(y) = \frac{1}{3}\left(1+\frac{5}{3}D'\right)y = \frac{1}{3}\left(y+\frac{5}{3}\right)$

Sub these in ①, we have

$$P.I = 4\left(\frac{1}{3}\right) + \cancel{3}\cdot\frac{1}{\cancel{3}}\left(x+\frac{4}{3}\right) + \overset{2}{\cancel{6}}\cdot\frac{1}{\cancel{3}}\left(y+\frac{5}{3}\right)$$

$$= \frac{4}{3} + x + \frac{4}{3} + 2y + \frac{10}{3}$$

$$P.I = x+2y+\frac{18}{3} = x+2y+6$$

$\therefore$ G.S is $z = C.F + P.I$

$$= e^{x}f_1(y-x) + e^{3x}f_2(y+2x) + x+2y+6.$$

---

⑧ $(D-3D'-2)^3 z = 6e^{2x}\sin(3x+y)$

Sol: Here $f(D,D') = \underset{\substack{\Downarrow \\ m=3,\ \alpha=2}}{(D-3D'-2)^3}$, repeated 3 times

$$C.F = e^{\alpha x}f_1(y+mx) + x e^{\alpha x}f_2(y+mx) + x^2 e^{\alpha x}f_3(y+mx)$$

$$\Rightarrow C.F = e^{2x}f_1(y+3x) + x e^{2x}f_2(y+3x) + x^2 e^{2x}f_3(y+3x).$$

$$P.I = \frac{1}{f(D,D')}Q(x,y)$$

$$= \frac{1}{(D-3D'-2)^3}\,6e^{2x}\sin(3x+y) \qquad \left(e^{ax+by}\,V\right)$$

put $D = D+a$, $D' = D'+b$, $a=2$, $b=0$

$$= 6e^{2x}\cdot\frac{1}{(D+\cancel{2}-3D'-\cancel{2})^3}\sin(3x+y)$$

$$= 6e^{2x}\frac{1}{(D-3D')^3}\sin(3x+y).$$ This is Homogeneous D.E of the form $\frac{1}{f(D,D')}\sin(ax+by)$.

Here $a=3$, $b=1$, n= order $=3$

$f(D,D') = f_1(3,1) = (3-3)^3 = 0$ ie $ax+by$ Combination fun

Case of failure.

$$P.I = 6e^{2x}\cdot x\cdot\frac{1}{\frac{\partial}{\partial D}(D-3D')^3}\sin(3x+y).$$

$$P.I = \cancel{6}^{2} x e^{2y} \frac{1}{\cancel{3}(D-3D')^2} \sin(3x+y)$$

put $D=3$, $D'=1$, $n=2$

$(D-3D')^2 = (3-3)^2 = 0$, Case of failure.

$$P.I = 2xe^{2y}\left\{x \frac{1}{\frac{\partial}{\partial D}(D-3D')^2} \sin(3x+y)\right\}$$

$$= \cancel{2}x^2e^{2x}\frac{1}{\cancel{2}(D-3D')}\sin(3x+y)$$

put $D=a=3$, $D'=b=1$, $n=1$

$D-3D' = 3-3=0$, again Case of failure.

$$P.I = x^2e^{2x}\left\{x.\frac{1}{\frac{\partial}{\partial D}(D-3D')}\sin(3x+y)\right\}$$

$$P.I = x^3e^{2x}\sin(3x+y)$$

G.S $z = C.F + P.I$

$$= e^{2y}f_1(y+3x) + xe^{2x}f_2(y+3x) + x^2e^{2y}f_3(y+3x) + x^3e^{2y}\sin(3x+y)$$

H.W

① $r-t+p-q=0$ A) $z = f_1(y+x) + e^{-x} f_2(y-x)$

② $(D+D'-1)(D+2D'-2)z=0$ A) $z = e^x f_1(y-x) + e^{2x} f_2(y-2x)$

③ $(D^2-D'^2-3D+3D')z = e^{x-2y}$

A) $z = f_1(y+x) + e^{3x} f_2(y-x) - \frac{1}{12} e^{x-2y}$

④ $(D^2-DD'+D'-1)z = \sin(x+2y)$

A) $z = e^x f_1(y) + e^{-x} f_2(y+x) - \frac{1}{2}\cos(x+2y)$

⑤ $(D-D'-1)(D-D'-2)z = e^{3x-y}+x$

A) $z = e^x f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{6} e^{3x-y} + \frac{1}{2}(x+3/2)$

(46)

classification of second order P.D.E

The second order linear P.D.E

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F(u) = 0$$

where A, B, C, D, E are real Constants is said to be

(i) Hyperbolic if $B^2 - 4AC > 0$

(ii) parabolic if $B^2 - 4AC = 0$

(iii) Elliptic if $B^2 - 4AC < 0$

for example.

(i) The eqn $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$ (special case of wave eqn)

is hyperbolic since $A = 1$, $B = 0$, $C = -1$,

$$B^2 - 4AC = 0 - 4(1)(-1) = 4 > 0$$

(ii) The eqn $\frac{\partial u}{\partial t} = c \frac{\partial^2 u}{\partial t^2}$ (heat eqn) is parabolic

since $A = c$, $B = 0$, $C = 0$ and $B^2 - 4AC = 0$

(iii) The eqn $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (two dimensional Laplace eqn) is elliptic, since $A = 1$, $B = 0$, $C = 1$

$$B^2 - 4AC = -4 < 0.$$

Applications of PDE.

Method of Separation of variables :-

When we have a partial differential eqn involving two independent variables say x and y, we seek a solution in the form $X(x)Y(y)$ and write down various types of solutions. The following example will explain the method.

㊼ problems on method of separation of variables

① solve $y^3\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y} = 0$ —①

sol: let $z = X(x)Y(y)$ be the soln of ①

$$\frac{\partial z}{\partial x} = X'(x)Y(y), \quad \frac{\partial z}{\partial y} = X(x)Y'(y)$$

sub these in ①, we have

$$y^3X'(x)Y(y) + x^2X(x)Y'(y) = 0$$

$$\Rightarrow y^3X'(x)Y(y) = -x^2X(x)Y'(y) = 0$$

$$\Rightarrow \frac{X'(x)}{x^2X(x)} = -\frac{Y'(y)}{y^3Y(y)}$$

In the above L.H.S is a function of x and R.H.S is a function of y and these are equal for all values of x and y. This is possible if and only if each is equal to the same constant (say) λ. This λ is called separation constant

∴ we have $\frac{X(x)}{x^2X(x)} = \frac{-Y'(y)}{y^3Y(y)} = \lambda$ —③

from ③, we get the two ordinary differential equations

$$X'(x) = \lambda x^2X(x) \quad \&$$

$$\Rightarrow \frac{X(x)}{\cancel{X(x)}} = \lambda x^2X(x)$$

$$\Rightarrow \frac{dX}{dx} = \lambda x^2X$$

$$\Rightarrow \frac{dX}{X} = \lambda x^2dx$$

Integrate on B.S

$$\int\frac{dx}{x} = \lambda\int x^2dx$$

$$\Rightarrow \log X = \lambda\frac{x^3}{3} + \log c_1$$

$$\Rightarrow \log\left(\frac{X}{c_1}\right) = \lambda\frac{x^3}{3}$$

$$Y'(y) = -\lambda y^3Y(y)$$

$$\frac{dY}{dy} = -\lambda y^3Y$$

$\Rightarrow \frac{dY}{Y} = -\lambda y^3dy$, Integrate on B.S

$$\Rightarrow \log Y = -\lambda\frac{y^4}{4} + \log c_2$$

$$\Rightarrow \log Y - \log c_2 = -\lambda\frac{y^4}{4}$$

$$\Rightarrow \log\left(\frac{Y}{c_2}\right) = -\lambda\frac{y^4}{4}$$

$$\Rightarrow \frac{Y}{c_2} = e^{-\lambda\frac{y^4}{4}}$$

$$\Rightarrow \frac{x}{c_1} = e^{\lambda \frac{x^3}{3}}, \quad \frac{y}{c_2} = e^{-\lambda \frac{y^4}{4}}$$

$$\Rightarrow X = c_1 e^{\lambda \frac{x^3}{3}}, \quad Y = c_2 e^{-\lambda \frac{y^4}{4}}$$

$\therefore$ soln of ① is given by $z = X(x)Y(y)$

$$\Rightarrow z = c_1 e^{\lambda \frac{x^3}{3}} . c_2 e^{-\lambda \frac{y^4}{4}}$$

$$z = c e^{\lambda \left(\frac{x^3}{3} - \frac{y^4}{4}\right)}$$

, $c = c_1 c_2$ is an arbitrary constant

② Solve by the method of separation of variables

$u_x = 2u_t + u$ where $u(x,0) = 6e^{-3x}$

Sol: Given P.D.E $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ —①

Let the soln of eqn ① using method of seperation of variables is

$$u = X(x)\,T(t)$$

$$\Rightarrow \frac{\partial u}{\partial x} = X'T, \quad \frac{\partial u}{\partial t} = XT'$$

sub these in ①, we have

$$X'T = 2XT' + XT$$

$$\Rightarrow X'T = X[2T' + T]$$

$$\Rightarrow \frac{X'}{X} = \frac{2T'+T}{T} = \lambda \text{ (separation constant)}$$

Now $\frac{X'}{X} = \lambda$

$$\Rightarrow X' = \lambda X$$

$$\Rightarrow \frac{dX}{dx} = \lambda X$$

$$\Rightarrow \frac{dX}{X} = \lambda dx$$

Variable separable, Integrate on B.S

$$\frac{2T'+T}{T} = \lambda$$

$$2T' + T = \lambda T$$

$$\Rightarrow 2T' = (\lambda - 1)T$$

$$\Rightarrow 2\frac{dT}{dt} = (\lambda - 1)T$$

$$\Rightarrow \frac{dT}{T} = \frac{(\lambda-1)}{2} dt$$

V.S, Integrate on B.S

㊾ $\int \frac{dX}{X} = \lambda \int dx.$

$\Rightarrow \log X = \lambda x + \log c_1$

$\Rightarrow \log\left(\frac{X}{c_1}\right) = \lambda x$

$\Rightarrow \frac{X}{c_1} = e^{\lambda x}$

$\Rightarrow \boxed{X(x) = c_1 e^{\lambda x}}$

$\int \frac{dT}{T} = \frac{(\lambda-1)}{2} \int dt$

$\Rightarrow \log T = \frac{(\lambda-1)}{2} t + \log c_2$

$\Rightarrow \log T - \log c_2 = \frac{(\lambda-1)}{2} t$

$\Rightarrow \log\left(\frac{T}{c_2}\right) = \frac{(\lambda-1)}{2} t$

$\Rightarrow \frac{T}{c_2} = e^{\frac{(\lambda-1)}{2} t}$

$\Rightarrow \boxed{T(t) = c_2 e^{\left(\frac{\lambda-1}{2}\right)t}}.$

$\therefore u(x,t) = X(x)T(t)$

$\Rightarrow u(x,t) = c_1 e^{\lambda x} . c_2 e^{\left(\frac{\lambda-1}{2}\right)t}$

$\Rightarrow u(x,t) = c \, e^{\lambda x + \left(\frac{\lambda-1}{2}\right)t}$ ② $(\because c_1 \cdot c_2 = c)$

but given $u(x,0) = 6e^{-3x}$

$\therefore u(x,0) = c \, e^{\lambda x}.$

$\Rightarrow 6e^{-3x} = c \, e^{\lambda x}$

Comparing like coefficients on B.S, we have.

$\boxed{c = 6, \lambda = -3.}$

Sub c, λ values in ② we get the sol'n of ①

$\boxed{u(x,t) = 6 \, e^{-3x-2t}}$

⑬ using method of separation of variables solve $u_{xt} = e^{-t} \cos x$ with $u(x,0)=0$ and $u(0,t)=0$

Sol: Given P.D.E $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$ —①

Let the sol'n of eqn ① using method of separation of variables is $u = X(x)T(t)$

(50)

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial t} = X^1 T^1 .$$

sub this in ①, we have

$X^1 T^1 = e^{-t} \cos x$. where $X^1 = \frac{dX}{dx}$, $T^1 = \frac{dT}{dt}$

$$\Rightarrow \frac{X^1}{\cos x} = \frac{e^{-t}}{T^1} = \lambda \text{ (separation constant)}$$

$\frac{X^1}{\cos x} = \lambda$

$\Rightarrow X^1 = \lambda \cos x$

$\Rightarrow \frac{dX}{dx} = \lambda \cos x$

$\Rightarrow dX = \lambda \cos x \, dx$
Integrate on B.S

$\Rightarrow \underline{X = \lambda \sin x + c_1}$

$\frac{e^{-t}}{T^1} = \lambda$

$\Rightarrow T^1 \lambda = e^{-t}$

$\Rightarrow T^1 = \frac{1}{\lambda} e^{-t}$ (v-s, $\int$ on B.S)

$\Rightarrow \frac{dT}{dt} = \frac{1}{\lambda} e^{-t}$

$\Rightarrow dT = \frac{1}{\lambda} e^{-t} dt$

$\Rightarrow \boxed{T = -\frac{1}{\lambda} e^{-t} + c_2}$

sub $X(x)$ & $T(t)$ values in ②, we have.

$$u(x,t) = [\lambda \sin x + c_1] \left[-\frac{1}{\lambda} e^{-t} + c_2\right] \text{ — ③}$$

but given $u(x,0) = 0$ & $u(0,t) = 0$.

Now $u(x,0) = 0$, from ③

$\Rightarrow$ ~~0~~ $u(x,0) = (\lambda \sin x + c_1)\left(\frac{-1}{\lambda} + c_2\right)$

$\Rightarrow 0 = (\lambda \sin x + c_1)\left(\frac{-1}{\lambda} + c_2\right)$

$\Rightarrow \frac{-1}{\lambda} + c_2 = 0 \Rightarrow \boxed{c_2 = \frac{1}{\lambda}}$ ($\because \lambda \sin x + c_1 \neq 0$)

Now $u(0,t) = 0$, from ③

$u(0,t) = c_1 \left[\frac{-1}{\lambda} e^{-t} + c_2\right]$

$\Rightarrow 0 = c_1 \left[\frac{-1}{\lambda} e^{-t} + c_2\right] \Rightarrow \boxed{c_1 = 0}$

sub c_1, c_2 values in ③ we get the sol of ①

$$u(x,t) = \lambda \sin x \left[\frac{-1}{\lambda} e^{-t} + \frac{1}{\lambda}\right] = \underline{\sin x (1 - e^{-t})}$$

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Code No: R19BS1203 **R19** SET - 1

I B. Tech II Semester Regular Examinations, December - 2020
MATHEMATICS-III

(Com. to CE, EEE, ECE, CSE, Chem E, EIE, IT, Auto E, Min E, Pet E)

Time: 3 hours Max. Marks: 75

Answer any five Questions one Question from Each Unit
All Questions Carry Equal Marks

~~~~~~

1. a) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at (1, -2, -1) in the direction 2i –j – 2k. (8M)

   b) If $\bar{r}$ is the position vector of the point (x, y, z), prove that $\nabla^2(r^n) = (n+1)r^{n-2}$. (7M)

Or

2. a) Use Gauss divergence theorem to evaluate $\iint_S (yz^2\bar{i} + zx^2\bar{j} + 2z^2\bar{k}).ds$, where S is the closed surface bounded by the xy - plane and the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ above this plane. (8M)

   b) Using Green's theorem evaluate $\int_C (2xy - x^2)dx + (x^2 + y^2)dy$ where C is the closed curve of the region bounded $y = x^2$ and $y^2 = x$. (7M)

3. a) Using Laplace transform evaluate $\int_0^\infty e^t t^2 \sin atdt$. (7M)

   b) Find $L^{-1}\left[\cot^{-1}\left(\frac{s+a}{b}\right)\right]$. (8M)

Or

4. a) Using convolution theorem, evaluate $L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^3}\right]$. (6M)

   b) Solve $(D^2 + 2D + 1)y = 3te^{-t}$ given that $y(0) = 4, y'(0) = 2$. (9M)

5. a) Find the Fourier series of $f(x) = \begin{cases} \frac{-1}{2}(\pi - x), \text{for } -\pi < x < 0 \\ \frac{1}{2}(\pi - x), \text{for } 0 < x < \pi \end{cases}$ (8M)

   b) Find the Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$. (7M)

Or

6. a) Find the half – range cosine series for the function $f(x) = (x-1)^2$ in the interval. Hence show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{8}$. (8M)

   b) Using Fourier integral, show that $\int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin x\lambda = \begin{cases} \frac{\pi}{2}, & if\ 0 < x < \pi \\ 0, & if\ x > \pi \end{cases}$. (7M)
~~~~~~

7. a) Form a partial differential equation by eliminating a and b from $\log(az-1) = x + ay + b$. (7M)

b) Solve $px(z-2y^2) = (z-qy)(z-y^2-2x^3)$. (8M)

Or

8. a) Solve $z^2(p^2+q^2) = x^2+y^2$. (7M)

b) Solve $\frac{x^2}{p} + \frac{y^2}{q} = z$. (8M)

9. a) Solve $\left(D^2 - 4DD' + 4D'^2\right)z = e^{2x+y}$. (8M)

b) Solve $\left(D^2 - 3DD' + 2D'^2\right)z = sin(x-2y)$. (7M)

Or

10. a) Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where $u(x,0) = 6e^{-3x}$ by method of separation of variables. (6M)

b) A tightly stretched string with fixed end points x = 0 and x = L is initially in a position given by $y = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position; find the displacement y(x, t). (9M)

Code No: R19BS1203 **R19** SET - 2

I B. Tech II Semester Regular Examinations, December - 2020
MATHEMATICS-III

(Com. to CE, EEE, ECE, CSE, Chem E, EIE, IT, Auto E, Min E, Pet E)

Time: 3 hours Max. Marks: 75

Answer any five Questions one Question from Each Unit
All Questions Carry Equal Marks

~~~~~~

1\. a) Find the angle of intersection of the spheres $x^2 + y^2 + z^2 = 39$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z + 52 = 0$ at the point (4, -3, 2). (8M)

b) Find the constants a, b, c so that $(x+2y+az)\bar{i}+(bx-3y-z)\bar{j}+(4x+cy+2z)\bar{k}$ is irrotational. Also find ϕ (scalar potential). (7M)

Or

2\. Verify Stoke's theorem for $\bar{F}=(x^2-y^2)\bar{i}+2xy\,\bar{j}$ over the box bounded by the planes x = 0, x = a, y = 0, y = b, z = 0, z = c. (15M)

3\. a) Find $L\left[e^{-3t}\int_0^t \frac{1-\cos t}{t^2}dt\right]$. (7M)

b) Find $L^{-1}\left[\frac{e^{-2s}}{s^2+4s+5}\right]$. (8M)

Or

4\. a) Solve $y'''-3y''+3y'-y=t^2e^t$ given that $y=1, y'=0, y''=-2$ at t = 0. (9M)

b) Using convolution theorem, evaluate $L^{-1}\left[\frac{s^2}{\left(s^2+a^2\right)^2}\right]$. (6M)

5\. a) Find the Fourier series for $f(x)=2x-x^2$ in $0<x<3$. (8M)

b) Find the Fourier cosine transform of $f(x)=\frac{e^{-ax}}{x}$. (7M)

Or

6\. a) Find the half – range cosine series for the function $f(x)=(x-1)^2$ in the interval. Hence show that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+....=\frac{\pi^2}{8}$. (8M)

b) Using Fourier integral, show that $e^{-x}\cos x=\frac{2}{\pi}\int_0^\infty \frac{\lambda^2+2}{\lambda^4+4}\cos\lambda x d\lambda$. (7M)

7\. a) Find the partial differential equation arising from $\phi(x+y+z, x^2+y^2+z^2)=0$. (7M)

b) Solve $(x^2-yz)p+(y^2-zx)q=z^2-xy$. (8M)

Or
~~~~~~

8. a) Solve $\left(\frac{p}{2}+x\right)^2+\left(\frac{q}{2}+y\right)^2=1$. (7M)

   b) Solve $(x+pz)^2+(y+qz)^2=1$. (8M)

9. a) Solve $\left(D^2+3DD'+D'^2\right)z=e^{2x+3y}$ (7M)

   b) Solve $\left(D^2+DD'-6D'^2\right)z=cos\left(3x+y\right)$. (8M)

Or

10. a) Solve $u_{xt}=e^{-t}\cos x$ with $u(x,0)=0$ and $u(0,t)=0$ by method of separation of variables. (6M)

    b) A rightly stretch of length 20 cms., fastened at both ends is displaced from its position of equilibrium, by imparting to each of its points an initial velocity given by: $V(x)=\begin{cases} x \;\;, & 0\le x\le 10 \\ 20-x, & 10\le x\le 20\end{cases}$. X being the distance from one end. Determine the displacement at any subsequent time. (9M)

Code No: R19BS1203 **R19** SET - 3

I B. Tech II Semester Regular Examinations, December - 2020

MATHEMATICS-III

(Com. to CE, EEE, ECE, CSE, Chem E, EIE, IT, Auto E, Min E, Pet E)

Time: 3 hours Max. Marks: 75

Answer any five Questions one Question from Each Unit

All Questions Carry Equal Marks

* * *

1\. a) Find the values of a and b so that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at the point (1, – 1, 2). (7M)

b) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at (1, -2, -1) in the direction 2i –j – 2k. (8M)

Or

2\. Verify Gauss divergence theorem for $\overline{F} = x^3\overline{i} + y^3\overline{j} + z^3\overline{k}$ taken over the cube bounded by x = 0, x = a, y = 0, y = a, z = 0, z = a. (15M)

3\. a) Find $L\left[e^{-3t}\int_0^t \frac{1-\cos t}{t^2}\,dt\right]$. (7M)

b) Find $L^{-1}\left[\frac{s+1}{s^2-2s+2}\right]$. (8M)

Or

4\. a) Find $L^{-1}\left\{\frac{s^2}{s^4+4a^4}\right\}$ using convolution theorem. (7M)

b) Solve $(D^2+6D+9)y = \sin t$ given that $y(0)=1, y'(0)=0$. (8M)

5\. a) Find the Fourier series of $f(x) = \begin{cases} 0, \text{for } -\pi < x < 0 \\ x^2, \text{for } 0 < x < \pi \end{cases}$. (9M)

b) If F(p) is the complex Fourier transform of f(x), then the complex Fourier transform of f(x) cos ax is $\frac{1}{2}[F(p+a)+F(p-a)]$. (6M)

Or

6\. a) Obtain half range Fourier sine series for e^x in $0 < x < 1$. (8M)

b) Find the finite Fourier sine transform of $f(x) = x^3$ in $(0, \pi)$. (7M)

7\. a) From the partial differential equation by eliminating the arbitrary functions f and g from $z = xf(ax+by) + g(ax+by)$. (6M)

b) Solve $(x^2-y^2-yz)p + (x^2-y^2-zx)q = z(x-y)$. (9M)

Or

8\. a) Solve $p^2x^2 + q^2y^2 = z^2$. (8M)

b) Solve $z^2(p^2+q^2) = x^2+y^2$. (7M)

9. a) Solve $(D+D')^2 z = e^{x-y}$. (7M)

b) Solve $(D^2 - D'^2)z = \cos 2x \cos 3y$. (8M)

Or

10. a) Solve $4u_x + u_y = 3u$ and $u(0, y) = e^{-5y}$ by method of separation of variables. (6M)

b) A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set to vibrate by giving each of its points a velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t. (9M)

Code No: R19BS1203 **R19** SET - 4

I B. Tech II Semester Regular Examinations, December - 2020

MATHEMATICS-III

(Com. to CE, EEE, ECE, CSE, Chem E, EIE, IT, Auto E, Min E, Pet E)

Time: 3 hours Max. Marks: 75

Answer any five Questions one Question from Each Unit
All Questions Carry Equal Marks

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1. a) Find the constants a, b, c so that $(x+2y+az)\bar{i}+(bx-3y-z)\bar{j}+(4x+cy+2z)\bar{k}$ is irrotational. Also find ϕ (scalar potential). (8M)

   b) Prove that div. (grad r^m) = m (m + 1) r^{m-2}. (7M)

Or

2. Verify Gauss divergence theorem for $\bar{F}=x^2\bar{i}+y^2\bar{j}+z^2\bar{k}$, over the cube formed by the planes x = 0, x = a, y = 0, y = b, z = 0, z = c. (15M)

3. a) Evaluate $\int_0^\infty \frac{\cos 6t-\cos 4t}{t}dt$ using Laplace transforms. (8M)

   b) Find the inverse Laplace transform of $\frac{s+1}{s^2-2s+2}$. (7M)

Or

4. a) Using convolution theorem, evaluate $L^{-1}\left\{\frac{s^2}{(s^2+a^2)^3}\right\}$. (6M)

   b) Using Laplace transform, solve $(D^2+5D-6)y = x^2e^{-x}$, y(0) = a, $y'(0)=b$. (9M)

5. a) Find the Fourier series of $f(x)=\left(\frac{\pi-x}{2}\right)^2$ in the interval $0<x<2\pi$. (8M)

   b) Using Fourier integral, show that $\int_0^\infty \frac{1-\cos\pi\lambda}{\lambda}\sin x\lambda = \begin{cases}\frac{\pi}{2}, & if\ 0<x<\pi \\ 0\ , & if\ x>\pi\end{cases}$. (7M)

Or

6. a) Find the Fourier cosine series for $f(x)=2x-x^2$, in $0<x<3$ (8M)

   b) Find the finite Fourier cosine transform of $f(x)=\sin ax$ in $(0,\pi)$. (7M)

7. a) Form the partial differential equation by eliminating the arbitrary function f from $xyz=f(x^2+y^2+z^2)$. (6M)

   b) Solve $x^2(z-y)p+y^2(x-z)q=z^2(y-x)$. (9M)

Or

8. a) Solve $q^2=z^2p^2(1-p^2)$. (8M)

   b) Solve $(x+pz)^2+(y+qz)^2=1$. (7M)
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9. a) Solve $\left(4D^2+12DD'+9D'^2\right)z=e^{3x-2y}$. (7M)

   b) Solve $\left(D^3-7DD'^2-6D'^3\right)z=\sin(x+2y)$. (8M)

Or

10. a) Solve $3u_x+2u_y=0$ with $u(x,0)=4e^{-x}$ by method of separation of variables. (6M)

    b) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y=y_0\sin^3\frac{\pi x}{l}$. If it is released from rest from this position, find displacement $y(x,t)$. (9M)

Code No: R19BS1203 **R19** SET - 1

I B. Tech II Semester Supplementary Examinations, July/August - 2021
MATHEMATICS-III
(Com. to CE, EEE, ECE, CSE, Chem. E, EIE, IT, Auto E, Min E, Pet E)

Time: 3 hours Max. Marks: 75

Answer any five Questions one Question from Each Unit
All Questions Carry Equal Marks

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1. a) ϕ, ψ be two scalar functions then prove that $div\left(\phi\nabla\psi\right)=\phi\nabla^2\psi+\nabla\phi.\nabla\psi$ (8M)

   b) Compute $\iint_s (a^2 x^2+b^2 y^2+c^2z^2)^{1/2}\, ds$ over the ellipsoid $ax^2+by^2+cz^2=1$. Using Gauss divergence theorem. (7M)

Or

2. a) Prove that $Curl\left(\overline{A}\times\overline{B}\right)=\overline{A}div\overline{B}-\overline{B}div\overline{A}+\left(\overline{B}.\nabla\right)\overline{A}-\left(\overline{A}.\nabla\right)\overline{B}$ (8M)

   b) Evaluate $\oint_C \overline{F}.d\overline{r}$ where $\overline{F}=xy\,\overline{i}+\left(x^2+y^2\right)\overline{j}$ and C is the axes from $x=2$ to $x=4$ and the line from y = 0 to y = 12. (7M)

3. a) Find $L\,\{e^t(t\ \ sinht)\}$ (7M)

   b) Using Convolution theorem find the inverse Laplace transform of $\frac{1}{s^2(1+s)^2}$ (8M)

Or

4. a) Solve the differential equations by using Laplace transforms method (8M)
   $(D^2+4D+3)y=e^{-t}\text{if}y(0)=1, y'(0)=1$

   b) Find Laplace transform of $f(t)$ where $f(t)=\begin{cases}e^t, & 0<t<1\\ 3, & t>1\end{cases}$ (7M)

5. a) Find the Half range sine series of f(x) = x in [0,π] (8M)
   Hence deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}......=\frac{\pi}{4}$

   b) Find the Fourier Cosine transform of the function $f(x)=\begin{cases}\sin ax & if\ \ x<a\\ 0 & if\ \ x>a\end{cases}$ (7M)

Or

6. a) Obtain the Fourier series of $f(x)=x^2-2 \quad -2\le x\le 2$ (8M)

   b) Using Fourier integral show that (7M)
   $$e^{-ax}-e^{-bx}=\frac{2(b^2-a^2)}{\pi}\int_0^{\infty}\frac{\lambda Sin\lambda x d\lambda}{\left(\lambda^2+a^2\right)\left(\lambda^2+b^2\right)}, a,b>0$$
~~~~~~

7\. a) Form the partial differential equation by eliminating arbitrary constants from $z = \left(\sqrt{x+a}\right)\left(\sqrt{y+b}\right)$ (8M)

b) Solve the PDE $y^2 p - xyq = x(z - 2y)$ (7M)

Or

8\. a) Solve the PDE $x^2 p^2 = yq^2$ (8M)

b) Form the partial differential equation by eliminating arbitrary function from $z = f(\sin x - \cos y)$ (7M)

9\. a) Solve the steady state equation of a rectangular plate of sides 'a' and 'b' insulated on the lateral surfaces subject to u(0, y) = 0 = u(a ,y) = u (x ,b) and u(x , 0) = x(a-x). (8M)

b) Solve the PDE $\left(D^2 - 7DD^1 + 12D^{1^2}\right)z = e^{x-y}$ (7M)

Or

10\. a) A tightly stretched string with fixed end points x = 0 and x = 1 is initially in a position given by $y = y_0 \sin^3 \frac{\pi}{1}$ if it is released from rest from this position, find displacement y(x, t). (8M)

b) Solve the PDE $\left(D^2 - DD^1\right)z = \sin x \cos 2y$ (7M)

Code No: R19BS1203 **R19** SET - 1

I B. Tech II Semester Regular Examinations, December - 2020

MATHEMATICS-III

(Com. to CE, EEE, ECE, CSE, Chem E, EIE, IT, Auto E, Min E, Pet E)

Time: 3 hours Max. Marks: 75

Answer any five Questions one Question from Each Unit
All Questions Carry Equal Marks

~~~~~~

1\. a) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at (1, -2, -1) in the direction 2i –j – 2k. (8M)

b) If $\bar{r}$ is the position vector of the point (x, y, z), prove that $\nabla^2(r^n) = (n+1)r^{n-2}$. (7M)

Or

2\. a) Use Gauss divergence theorem to evaluate $\iint_S (yz^2\bar{i} + zx^2\bar{j} + 2z^2\bar{k}).ds$, where S is the closed surface bounded by the xy - plane and the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ above this plane. (8M)

b) Using Green's theorem evaluate $\int_C (2xy - x^2)dx + (x^2 + y^2)dy$ where C is the closed curve of the region bounded $y = x^2$ and $y^2 = x$. (7M)

3\. a) Using Laplace transform evaluate $\int_0^\infty e^t t^2 \sin at dt$. (7M)

b) Find $L^{-1}\left[\cot^{-1}\left(\frac{s+a}{b}\right)\right]$. (8M)

Or

4\. a) Using convolution theorem, evaluate $L^{-1}\left[\frac{s}{(s^2+a^2)^3}\right]$. (6M)

b) Solve $(D^2 + 2D + 1)y = 3te^{-t}$ given that $y(0) = 4, y'(0) = 2$. (9M)

5\. a) Find the Fourier series of $f(x) = \begin{cases} \frac{-1}{2}(\pi - x), \text{for } -\pi < x < 0 \\ \frac{1}{2}(\pi - x), \text{for } 0 < x < \pi \end{cases}$ (8M)

b) Find the Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$. (7M)

Or

6\. a) Find the half – range cosine series for the function $f(x) = (x-1)^2$ in the interval. Hence show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{8}$. (8M)

b) Using Fourier integral, show that $\int_0^\infty \frac{1-\cos\pi\lambda}{\lambda}\sin x\lambda = \begin{cases} \frac{\pi}{2}, \ if\ 0 < x < \pi \\ 0\ ,\ if\ x > \pi \end{cases}$. (7M)
~~~~~~

7. a) Form a partial differential equation by eliminating a and b from $\log(az-1) = x + ay + b.$ (7M)

b) Solve $px(z-2y^2) = (z-qy)(z-y^2-2x^3)$. (8M)

Or

8. a) Solve $z^2(p^2+q^2) = x^2 + y^2$. (7M)

b) Solve $\dfrac{x^2}{p} + \dfrac{y^2}{q} = z$. (8M)

9. a) Solve $\left(D^2 - 4DD' + 4D'^2\right)z = e^{2x+y}$. (8M)

b) Solve $\left(D^2 - 3DD' + 2D'^2\right)z = sin(x-2y)$. (7M)

Or

10. a) Solve $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial t} + u$ where $u(x,0) = 6e^{-3x}$ by method of separation of variables. (6M)

b) A tightly stretched string with fixed end points x = 0 and x = L is initially in a position given by $y = y_0 \sin^3 \dfrac{\pi x}{l}$. If it is released from rest from this position; find the displacement y(x, t). (9M)

Code No: R19BS1203 **R19** SET - 2

I B. Tech II Semester Regular Examinations, December - 2020
MATHEMATICS-III
(Com. to CE, EEE, ECE, CSE, Chem E, EIE, IT, Auto E, Min E, Pet E)

Time: 3 hours Max. Marks: 75

Answer any five Questions one Question from Each Unit
All Questions Carry Equal Marks

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1\. a) Find the angle of intersection of the spheres $x^2 + y^2 + z^2 = 39$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z + 52 = 0$ at the point (4, -3, 2). (8M)

b) Find the constants a, b, c so that $(x+2y+az)\bar{i}+(bx-3y-z)\bar{j}+(4x+cy+2z)\bar{k}$ is irrotational. Also find ϕ (scalar potential). (7M)

Or

2\. Verify Stoke's theorem for $\bar{F}=(x^2-y^2)\bar{i}+2xy\,\bar{j}$ over the box bounded by the planes x = 0, x = a, y = 0, y = b, z = 0, z = c. (15M)

3\. a) Find $L\left[e^{-3t}\int_0^t \frac{1-\cos t}{t^2}dt\right]$. (7M)

b) Find $L^{-1}\left[\frac{e^{-2S}}{s^2+4s+5}\right]$. (8M)

Or

4\. a) Solve $y''' - 3y'' + 3y' - y = t^2e^t$ given that $y = 1, y' = 0, y'' = -2$ at t = 0. (9M)

b) Using convolution theorem, evaluate $L^{-1}\left[\frac{s^2}{\left(s^2+a^2\right)^2}\right]$. (6M)

5\. a) Find the Fourier series for $f(x)=2x-x^2$ in $0<x<3$. (8M)

b) Find the Fourier cosine transform of $f(x)=\frac{e^{-ax}}{x}$. (7M)

Or

6\. a) Find the half – range cosine series for the function $f(x)=(x-1)^2$ in the interval. Hence show that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+....=\frac{\pi^2}{8}$. (8M)

b) Using Fourier integral, show that $e^{-x}\cos x=\frac{2}{\pi}\int_0^\infty \frac{\lambda^2+2}{\lambda^4+4}\cos\lambda x d\lambda$. (7M)

7\. a) Find the partial differential equation arising from $\phi(x+y+z, x^2+y^2+z^2)=0$. (7M)

b) Solve $(x^2-yz)p+(y^2-zx)q=z^2-xy$. (8M)

Or
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8. a) Solve $\left(\frac{p}{2}+x\right)^2+\left(\frac{q}{2}+y\right)^2=1$. (7M)

   b) Solve $(x+pz)^2+(y+qz)^2=1$. (8M)

9. a) Solve $\left(D^2+3DD'+D'^2\right)z=e^{2x+3y}$ (7M)

   b) Solve $\left(D^2+DD'-6D'^2\right)z=cos\left(3x+y\right)$. (8M)

Or

10. a) Solve $u_{xt}=e^{-t}\cos x$ with $u(x,0)=0$ and $u(0,t)=0$ by method of separation of variables. (6M)

    b) A rightly stretch of length 20 cms., fastened at both ends is displaced from its position of equilibrium, by imparting to each of its points an initial velocity given by: $V(x)=\begin{cases} x \quad , 0\le x\le 10 \\ 20-x, 10\le x\le 20 \end{cases}$. X being the distance from one end. Determine the displacement at any subsequent time. (9M)

Code No: R19BS1203 **R19** SET - 3

I B. Tech II Semester Regular Examinations, December - 2020

MATHEMATICS-III

(Com. to CE, EEE, ECE, CSE, Chem E, EIE, IT, Auto E, Min E, Pet E)

Time: 3 hours Max. Marks: 75

Answer any five Questions one Question from Each Unit
All Questions Carry Equal Marks

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1. a) Find the values of a and b so that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at the point (1, – 1, 2). (7M)

   b) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at (1, -2, -1) in the direction 2i –j – 2k. (8M)

Or

2. Verify Gauss divergence theorem for $\overline{F} = x^3\overline{i} + y^3\overline{j} + z^3\overline{k}$ taken over the cube bounded by x = 0, x = a, y = 0, y = a, z = 0, z = a. (15M)

3. a) Find $L\left[e^{-3t}\int_0^t \frac{1-\cos t}{t^2}dt\right]$. (7M)

   b) Find $L^{-1}\left[\frac{s+1}{s^2-2s+2}\right]$. (8M)

Or

4. a) Find $L^{-1}\left\{\frac{s^2}{s^4+4a^4}\right\}$ using convolution theorem. (7M)

   b) Solve $(D^2+6D+9)y = \sin t$ given that $y(0)=1, y'(0)=0$. (8M)

5. a) Find the Fourier series of $f(x) = \begin{cases} 0, \text{for } -\pi < \text{x} < 0 \\ \text{x}^2, \text{for } 0 < \text{x} < \pi \end{cases}$. (9M)

   b) If F(p) is the complex Fourier transform of f(x), then the complex Fourier transform of f(x) cos ax is $\frac{1}{2}[F(p+a)+F(p-a)]$. (6M)

Or

6. a) Obtain half range Fourier sine series for e^x in $0 < x < 1$. (8M)

   b) Find the finite Fourier sine transform of $f(x) = x^3$ in $(0, \pi)$. (7M)

7. a) From the partial differential equation by eliminating the arbitrary functions f and g from $z = xf(ax+by) + g(ax+by)$. (6M)

   b) Solve $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x-y)$. (9M)

Or

8. a) Solve $p^2x^2 + q^2y^2 = z^2$. (8M)

   b) Solve $z^2(p^2+q^2) = x^2 + y^2$. (7M)
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9. a) Solve $(D+D')^2 z = e^{x-y}$. (7M)

b) Solve $(D^2 - D'^2)z = \cos 2x \cos 3y$. (8M)

Or

10. a) Solve $4u_x + u_y = 3u$ and $u(0, y) = e^{-5y}$ by method of separation of variables. (6M)

b) A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set to vibrate by giving each of its points a velocity $\lambda x(l-x)$, find the displacement of the string at any distance x from one end at any time t. (9M)

Code No: R19BS1203 **R19** SET - 4

I B. Tech II Semester Regular Examinations, December - 2020
MATHEMATICS-III
(Com. to CE, EEE, ECE, CSE, Chem E, EIE, IT, Auto E, Min E, Pet E)

Time: 3 hours Max. Marks: 75

Answer any five Questions one Question from Each Unit
All Questions Carry Equal Marks

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1. a) Find the constants a, b, c so that $(x+2y+az)\bar{i}+(bx-3y-z)\bar{j}+(4x+cy+2z)\bar{k}$ is irrotational. Also find ϕ (scalar potential). (8M)

   b) Prove that div. (grad r^m) = m (m + 1) r^{m-2}. (7M)

Or

2. Verify Gauss divergence theorem for $\bar{F}=x^2\bar{i}+y^2\bar{j}+z^2\bar{k}$, over the cube formed by the planes x = 0, x = a, y = 0, y = b, z = 0, z = c. (15M)

3. a) Evaluate $\int_0^\infty \frac{\cos 6t-\cos 4t}{t}dt$ using Laplace transforms. (8M)

   b) Find the inverse Laplace transform of $\frac{s+1}{s^2-2s+2}$. (7M)

Or

4. a) Using convolution theorem, evaluate $L^{-1}\left\{\frac{s^2}{(s^2+a^2)^3}\right\}$. (6M)

   b) Using Laplace transform, solve $(D^2+5D-6)y = x^2 e^{-x}$, y(0) = a, $y'(0)=b$. (9M)

5. a) Find the Fourier series of $f(x)=\left(\frac{\pi-x}{2}\right)^2$ in the interval $0<x<2\pi$. (8M)

   b) Using Fourier integral, show that $\int_0^\infty \frac{1-\cos\pi\lambda}{\lambda}\sin x\lambda = \begin{cases}\frac{\pi}{2}, & if\ 0<x<\pi\\ 0\ , & if\ x>\pi\end{cases}$. (7M)

Or

6. a) Find the Fourier cosine series for $f(x)=2x-x^2$, in $0<x<3$ (8M)

   b) Find the finite Fourier cosine transform of $f(x)=\sin ax$ in $(0,\pi)$. (7M)

7. a) Form the partial differential equation by eliminating the arbitrary function f from $xyz=f(x^2+y^2+z^2)$. (6M)

   b) Solve $x^2(z-y)p+y^2(x-z)q=z^2(y-x)$. (9M)

Or

8. a) Solve $q^2=z^2p^2(1-p^2)$. (8M)

   b) Solve $(x+pz)^2+(y+qz)^2=1$. (7M)
~~~~~~

9\. a) Solve $(4D^2+12DD'+9D'^2)z=e^{3x-2y}$. (7M)

b) Solve $(D^3-7DD'^2-6D'^3)z=\sin(x+2y)$. (8M)

Or

10\. a) Solve $3u_x+2u_y=0$ with $u(x,0)=4e^{-x}$ by method of separation of variables. (6M)

b) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y=y_0\sin^3\frac{\pi x}{l}$. If it is released from rest from this position, find displacement $y(x,t)$. (9M)

II B. Tech I Semester Regular Examinations, March - 2021
MATHEMATICS - III
(Agricultural Engineering)

Time: 3 hours Max. Marks: 75

Answer any **FIVE** Questions each Question from each unit
All Questions carry **Equal** Marks
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1 a) Evaluate Curl of $\bar{V} = e^{xyz}\left(i + j + k\right)$ at the point $(1,\ 2,\ \ 3)$. [8M]

b) Find the total work done in moving a particle in the force field [7M]
$\bar{F} = 3xy\ \bar{i} - 5z\ \bar{j} + 10x\ \bar{k}$ along the curve
$x = t^2 + 1,\ y = 2t^2\ ,z = t^3\ \ from\ t = 1\ to\ t = 2$.

Or

2 State Green's theorem and Verify Green's theorem in plane for
$\oint_C \left[\left(3x^2 - 8y^2\right)dx + \left(4y - 6xy\right)dy\right]$, where C is boundary of the region
defined by $y = \sqrt{x}\ \ and\ \ \ y = x^2$.

3 a) Find i) $L\left\{e^{-2t}\sin^3 t\right\}$ ii) $L\left\{t\cos at\right\}$ [6M]

b) Find i) $L^{-1}\left\{\dfrac{8s+20}{s^2-12s+32}\right\}$, ii) $L^{-1}\left\{\cot^{-1}\left(\dfrac{s+3}{2}\right)\right\}$. [4M]

c) Using Laplace transform, solve $y'' - 2y' - 8y = 0\ ,\ y(0) = 3,\ y'(0) = 6$. [5M]

Or

4 a) Find i) $L\left\{\cos^3 2t\right\}$ ii) $L\left\{\sinh at.\sin at\right\}$. [6M]

b) Find $L^{-1}\left\{\dfrac{1}{(s-a)(s-b)}\right\}$ using Convolution theorem. [4M]

c) Using Laplace transform, solve $y'' + y = 2e^t\ ,\ y(0) = 0,\ y'(0) = 2$. [5M]

5 a) Obtain the Fourier series of $f(x) = e^{ax}$ in the interval $\left(0, 2\pi\right)$. [8M]

b) Find the Fourier transform of $f(x) = \begin{cases} 1 & , for\ |x| < a \\ 0 & , for\ |x| > a \end{cases}$ [7M]

and hence deduce that $\int_0^\infty \dfrac{\sin ax}{x}dx$.

Or

6 a) Obtain the Fourier series for $f(x) = \begin{cases} -\pi\ , -\pi < x \le 0 \\ \ \ x,\ \ 0 < x \le \pi \end{cases}$. [8M]

Deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \ \cdots\cdots\ = \dfrac{\pi^2}{8}$.

b) Find the Fourier sine integral of $f\left(x\right) = e^{-ax} - e^{-bx}\ ,\ a > 0, b > 0$. [7M]
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7 a) Derive a partial differential equation from $xyz = f(x+y+z)$ [5M]

b) Solve the partial differential equation $p+3q = 5z+\tan(y-3x)$. [5M]

c) Solve the partial differential equation $p^2z^2+q^2=1$. [5M]

Or

8 a) Derive the partial differential equation by eliminating the constants from the equation $z = ax+by+a^2+b^2$. [5M]

b) Solve the partial differential equation $xp+yq=z$. [5M]

c) Solve the partial differential equation $p^2+q^2=m^2$. [5M]

9 a) Solve $\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}=\sin x.\cos 2y$. [8M]

b) Solve by the method of separation of variables $y^3z_x+x^2z_y=0$. [7M]

Or

10 A string of length L is initially at rest in equilibrium position and each of its points is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0}=bx(L-x)$ Find displacement $y(x,t)$. [15M]